

NUCLEAR POTENTIALS

IN

QUANTUM FIELD THEORY

The work in the following pages is declared to be original except where explicit references are made.

It is a pleasure to thank Professor N. Kemmer for the hospitality of the Felt Institute of Mathematical Physics and members of the staff thereof for revealing the subject; in particular Dr. D.J. Sandlin who suggested the field of my enquiry.

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by

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## PREFACE

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$$H|\Phi\rangle \equiv (H_0 + H')|\Phi\rangle = \frac{d}{dt}|\Phi\rangle$$

in Q.F.T. and use the eigenstates  $|\Phi\rangle$  of the free field equation

$$H_0|\Phi\rangle = \frac{d}{dt}|\Phi\rangle$$

## CHAPTER I

### A BRIEF SURVEY OF THE LITERATURE

The search for a satisfactory nuclear potential in Quantum Field Theory (hereafter referred to as Q.F.T.) has been long and the attempts have been numerous. The difficulty of the problem is twofold: firstly, because a potential is not Lorentz covariant, we have to find a way to convert the relativistic formalism of Q.F.T. into its non-relativistic equivalent; secondly, because no satisfactory method of calculating quantities appearing in Q.F.T. has yet evolved, we have to find ways to approximately calculate the potential that emerges out of the dissolution of the first difficulty. Different authors took different attitudes to either of these problems, and the result was a proliferation of proposals with varying degrees of success. Diverse though these proposals are, they fall into four main classes. We will briefly summarize them below. (We will work with the system of units in which  $c = \hbar = 1$ ).

#### (1) Canonical Transformation:-

We look at the Schrodinger equation

$$H|\Psi\rangle \equiv (H_0 + H')|\Psi\rangle = i\frac{d}{dt}|\Psi\rangle$$

This procedure can be repeated ad infinitum to remove the  $(a, j)$  in Q.F.T. and use the eigenvectors  $|\Phi\rangle$  of the free field elements of successively higher order in  $g$  and eventually we have, equation

$$H_0|\Phi\rangle = i\frac{d}{dt}|\Phi\rangle \quad \checkmark$$



as a basis of representation, of which the two-nucleon subset is labelled by  $\alpha, \beta, \dots$  with energies  $E_\alpha, E_\beta, \dots$  and the rest by  $j, k, \dots$  with energies  $E_j, E_k, \dots$ . Assuming that  $H'$  contains only terms of first order in the coupling constant  $g$ , the procedure is to find a unitary operator  $\exp i S^{(1)}$  so that

$$|\Psi'\rangle = e^{i S^{(1)}} |\Psi\rangle,$$

and 
$$e^{-i S^{(1)}} H e^{i S^{(1)}} = H + i[H, S^{(1)}] - \frac{1}{2}[[H, S^{(1)}], S^{(1)}] + \dots \quad (1.1)$$

to first order in  $g$  contains no terms coupling the two-nucleon subset to the rest of the Hilbert space. Since in (1.1) terms up to first order in  $g$  are

$$H_0 + H' + i[H_0, S^{(1)}]$$

our object is accomplished if

$$S_{\alpha j}^{(1)} = \frac{i H'_{\alpha j}}{E_\alpha - E_j}; \quad (1.2)$$

the other matrix elements of  $S^{(1)}$  are arbitrary. Notice (1.2) is not defined when  $E_\alpha = E_j$ .

The  $(\alpha, j)$  elements remaining in (1.1) are now at least of the second order in  $g$ , which we can remove in exactly the same way by suitably choosing a second unitary transformation  $\exp i S^{(2)}$ . This procedure can be repeated ad infinitum to remove the  $(\alpha, j)$  elements of successively higher order in  $g$  and eventually we have, at least in theory,

Eliminating  $(1-W)|\Psi\rangle$  from (1.4) and (1.5b), we get

$$(H_0 + V) = U^\dagger H U,$$

with

$$U = \exp S^{(1)} \exp S^{(2)} \dots \exp S^{(n)} \dots$$

and  $V$  having no  $(\alpha, j)$  elements. The two-nucleon subset can now be handled separately and a two-nucleon Schrodinger equation is then obtained.

In practice we stop after removing the  $(\alpha, j)$  elements to fourth order in  $g$ , and neglect terms of higher order. This method has been used, among others, by D. Feldman.<sup>(1)</sup>

Apart from the difficulty arising when  $E_\alpha = E_j$ , mentioned earlier, this method leaves no room for renormalization.

## (2) The Projection Operator:-

(1.6a) looks like a Schrodinger equation for a two-nucleon system if we regard  $W|\Psi\rangle$  as the state vector. We may not do so, however. The reason is that since eigenstates of the free Hamiltonian to form a basis of representation, but for the separation of the two-nucleon subset we use the projection operator  $W$  onto this subset. The stationary Schrodinger equation

$$H|\Psi\rangle = E|\Psi\rangle \quad (1.3)$$

can then be split into two equations:

$$(E - H_0)W|\Psi\rangle = WH'H|\Psi\rangle + WH'(1-W)|\Psi\rangle, \quad (1.4)$$

and

$$(1-W)(H_0 + H')|\Psi\rangle = E(1-W)|\Psi\rangle, \quad (1.5a)$$

or

$$(1-W)|\Psi\rangle = - \frac{1}{1 - \frac{1}{E - H_0} (1-W)H'} \frac{1}{E - H_0} (1-W)H'W|\Psi\rangle \quad (1.5b)$$

Eliminating  $(1-W)|\Psi\rangle$  from (1.4) and (1.5b), we get

$$(E-H_0)W|\Psi\rangle = \left[ WH' \frac{1}{1 - \frac{1}{E-H_0}(1-W)H'} W \right] W|\Psi\rangle \quad (1.6a)$$

$$\text{with } V = W(J^\dagger J)^N W(H_0 + H'J)W(J^\dagger J)^N W - WH_0W; \quad (1.11)$$

$$\equiv WH'JW|\Psi\rangle. \quad (1.6b)$$

From (1.5a) we get

$$|\Psi\rangle = \frac{1}{1 - \frac{1}{E-H_0}(1-W)H'} W|\Psi\rangle \quad (1.7a)$$

$$\equiv JW|\Psi\rangle. \quad (1.7b)$$

(1.6a) looks like a Schrodinger equation for a two-nucleon system if we regard  $W|\Psi\rangle$  as the state vector. We may not do so, however. The reason is that since

$$\langle\Psi_\alpha|\Psi_\beta\rangle = \langle\Psi_\alpha|W[W^2J^\dagger JW^2]W|\Psi_\beta\rangle = \delta_{\alpha\beta}, \quad (1.8)$$

$W|\Psi\rangle$  will change its norm when  $H'$  is switched off. This means that the "interaction term" in (1.6a) is not Hermitian. To avoid this difficulty Fukuda et al. proposed to use as the state vector of two interacting nucleons

$$|\chi\rangle = (W^2J^\dagger JW^2)^{1/2}|\Psi\rangle \equiv W(J^\dagger J)^{1/2}W|\Psi\rangle \quad (1.9)$$

instead of  $W|\Psi\rangle$ . Then from (1.6b) we get

$$E W(J^\dagger J)^{1/2}W|\Psi\rangle = W(J^\dagger J)^{1/2}[WH_0 + WH'J]W(J^\dagger J)^{1/2}W^2(J^\dagger J)^{1/2}W|\Psi\rangle,$$

or 
$$E|\chi\rangle = W(J^\dagger J)^{1/2}W(H_0 + H'J)W(J^\dagger J)^{1/2}W|\chi\rangle.$$

Hence we get

(3) The Bethe-Salpeter equation:-

$$(E - H_0) |\chi\rangle = V |\chi\rangle, \quad (1.10)$$

function of a two-nucleon state the expression

with 
$$V = W(J^\dagger J)^{1/2} W(H_0 + H'J) W(J^\dagger J)^{1/2} W - W H_0 W; \quad (1.11)$$

i.e., a stationary Schrodinger equation for the two-nucleon system.

The disadvantages of this method are, firstly, that in the complicated formal manipulation of the various operators we do not quite know the validity of the steps (in actual calculation when we expand formally the expression (1.11) we again come up with vanishing denominators as in the previous method); and secondly, there is again no provision for renormalization.

Since Haag<sup>(3)</sup> proved his famous theorem, that in Q.F.T. two free fields and two interacting fields cannot be connected by a unitary transformation, the greatest drawback of the two methods so far outlined is precisely the assumption of such a transformation. True, when we come to calculate anything in Q.F.T. we usually must fall back on perturbation theory, but at least we should try to have the definition of any physically meaningful quantity independent of the validity of the interaction picture. In the two methods we are about to describe, this criterion is satisfied.



(3) The Bethe-Salpeter equation:-

Gell-Mann and Low<sup>(4)</sup> showed that if we take as the wave function of a two-nucleon state the expression

$$\langle 0 | T (\psi(x_1) \psi(x_2)) | \Psi \rangle = \Psi(x_1, x_2)$$

where  $\psi(x_1)$ ,  $\psi(x_2)$  are dressed field operators in the Heisenberg picture and  $|\Psi\rangle$  is any of a complete set of state vectors, then  $\Psi(x_1, x_2)$  obeys the Bethe-Salpeter equation

$$\begin{aligned} & (\partial^{(1)} \cdot \gamma^{(1)} + m) (\partial^{(2)} \cdot \gamma^{(2)} + m) \Psi(x_1, x_2) \\ & = \int d^4 x'_1 d^4 x'_2 G(x_1, x_2; x'_1, x'_2) \Psi(x'_1, x'_2), \end{aligned} \quad (1.12)$$

or the two simultaneous integro-differential equations

$$(\partial^{(i)} \cdot \gamma^{(i)} + m) \Psi(x_1, x_2) = \int d^4 x'_1 d^4 x'_2 G^{(i)}(x_1, x_2; x'_1, x'_2) \Psi(x'_1, x'_2) \quad (1.13)$$

where the index  $i = 1, 2$ .

Krolikowski and Rzewuski<sup>(5)</sup> showed that (1.13) can be transformed into

$$(\partial^{(i)} \cdot \gamma^{(i)} + m) \Psi(x_1, x_2) = \int_{\sigma_1} \int_{\sigma_2} A^{(i)}(\sigma_1, \sigma_2) \gamma_{\mu}^{(1)} \gamma_{\nu}^{(2)} \Psi(x'_1, x'_2) d\sigma_{\mu}^{(1)} d\sigma_{\nu}^{(2)} \quad (1.14)$$

where  $A^{(i)}(\sigma_1, \sigma_2)$  is a function of  $x_1, x_2, x'_1, x'_2$  as well as a functional of the two space-like surfaces  $\sigma_1, \sigma_2$ , and must satisfy a certain equation (see reference 5). In particular, if we choose  $\sigma_1$  and  $\sigma_2$  to be the same plane with  $t = \text{constant}$ , we get

$$\begin{aligned} i \frac{\partial}{\partial t} \Psi(x_1, x_2, t) &= (H_0^{(1)} + H_0^{(2)}) \Psi(x_1, x_2, t) - \\ &- \iint d^3 x'_1 d^3 x'_2 V(x_1, x_2; x'_1, x'_2; t) \Psi(x'_1, x'_2, t), \end{aligned} \quad (1.15)$$



where

$$H_0^{(i)} = \beta^{(i)} (\underline{x}^{(i)} \cdot \nabla^{(i)} + m) = -i \underline{\alpha}^{(i)} \cdot \nabla^{(i)} + m \beta^{(i)},$$

$$V(x_1, x_2; x'_1, x'_2; t) = i [\beta^{(1)} A^{(1)}(x_1, x_2; x'_1, x'_2; t) + \beta^{(2)} A^{(2)}(x_1, x_2; x'_1, x'_2; t)] \beta^{(1)} \beta^{(2)}. \quad (1.16)$$

If the interaction is time-independent,  $G^{(i)}$  will depend on the  $x_i^0$  through  $(x_1^0 - x_2^0)$  and  $(x_1'^0 - x_2'^0)$  and

$$A^{(i)}(\sigma, \sigma) = A^{(i)}(x_1, x_2; x'_1, x'_2).$$

Although we have now in (1.15) a form of Hamiltonian formulation, it may include pair creation in which case we do not have an exclusively two-particle theory. The way to get round this difficulty is well-known: we simply use one or a series of Foldy-Wouthuysen transformations<sup>(18)</sup> to eliminate the coupling to the 'negative energy' states.

It should be pointed out that the majority of the authors, e.g. Klein and McCormick<sup>(6)</sup>, did not use the formulation outlined here which seems to the present author to be by far the most elegant.

The most serious theoretical difficulty involved in the use of the Bethe-Salpeter equation is that the kernel in (1.12) or (1.13), if calculated in perturbation theory, turns out to be divergent even in renormalizable theories<sup>(7)</sup>.

(4) The Method of the S-matrix:-

This method is the oldest, and many authors, e.g., Henley and Ruderman<sup>(8)</sup>, Hida et al.<sup>(9)</sup>, used it. Essentially it consists

in calculating the S-matrix elements (i.e., the transition amplitudes) for nucleon-nucleon scattering elements from Q.F.T., identifying them as those of a potential theory, and substituting them into the 'second' Lippmann-Schwinger equation (2.9a). Since so far the S-matrix elements in Q.F.T. are unambiguously defined only when the energies of the initial and final states are equal (on the energy-shell), whereas (2.9a) involves elements between states of different energies (off the energy-shell), a lot of arbitrariness is involved resulting from a choice for the latter elements. This arbitrariness is reflected in the potential obtained; an unfortunate choice may lead to divergences.

Another approach involving the use of dispersion relations avoids this difficulty at least as far as the on-the-energy-shell elements of the potential is concerned, though the philosophy of identifying the S-matrix elements is still retained. The most elegant formulation of this approach was put forward by Charap and Fubini<sup>(10), (11)</sup>. It is probably the best method to date; we will therefore discuss it more fully in Chapter 3.

Unfortunately their success was confined to boson-boson scattering only. Besides, they had to use the single representation, if not Mandelstam's conjecture of a double representation, of the scattering amplitude. The latter of these representations has not been proved for boson-boson scattering, while neither has been proved for the case of nucleon-nucleon scattering. In view of this we shall try in Chapter 4 to find a way of defining the potential, assuming only the validity of axiomatic Q.F.T., first put forward

systematically by Lehmann, Symanzik and Zimmermann. In the calculation of this potential, we are forced to use perturbation theory, which is done in Chapter 5. Discussion of our result is reported in Chapter 6. In the next chapter we shall prepare the ground for our definition of the potential by writing out in detail the Low equation which holds in potential theories.

in full for later use.

We consider an elastic scattering system in which  $H_0$  and  $V$  are respectively the free Hamiltonian and the potential. Let  $|\bar{\Psi}_\alpha\rangle$  satisfy

$$H_0 |\bar{\Psi}_\alpha\rangle = E_\alpha |\bar{\Psi}_\alpha\rangle$$

where we use a single suffix  $\alpha$  to denote the eigenvalues of observables forming a complete set. Lippmann and Schwinger<sup>(12)</sup> proved that a solution, with the same eigenvalues  $\alpha$ , to the equation

$$(H_0 + V) |\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle \quad (2.1)$$

can be written

$$|\Psi_\alpha \text{ in}\rangle = |\bar{\Psi}_\alpha\rangle - \frac{1}{H_0 - E_\alpha - i\epsilon} V |\Psi_\alpha \text{ in}\rangle \quad (2.2a)$$

or

$$|\Psi_\alpha \text{ out}\rangle = |\bar{\Psi}_\alpha\rangle - \frac{1}{H_0 - E_\alpha + i\epsilon} V |\Psi_\alpha \text{ out}\rangle \quad (2.2b)$$

where "in" ("out") denotes the boundary condition that in the infinite past (future) the state  $|\Psi_\alpha \text{ in}\rangle$  ( $|\Psi_\alpha \text{ out}\rangle$ ) behaves like a free state, i.e., the particles in the state do not interact.

Of course bound states, if they exist, cannot be represented in this way; they obey CHAPTER 2

# THE LIPPMANN SCHWINGER EQUATIONS AND THE LOW EQUATIONS

The material presented here is well-known and can be found in most text-books in one form or another. We write it out below in full for later use.

We consider an elastic scattering system in which  $H_0$  and  $V$  are respectively the free Hamiltonian and the potential. Let

$|\Phi_\alpha\rangle$  satisfy

$$H_0|\Phi_\alpha\rangle = E_\alpha|\Phi_\alpha\rangle$$

where we use a single suffix  $\alpha$  to denote the eigenvalues of observables forming a complete set. Lippmann and Schwinger<sup>(12)</sup> proved that a solution, with the same eigenvalues  $\alpha$ , to the equation

$$(H_0 + V)|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle \quad (2.1)$$

can be written

$$|\Psi_\alpha^{in}\rangle = |\Phi_\alpha\rangle - \frac{1}{H_0 - E_\alpha - i\varepsilon} V |\Psi_\alpha^{in}\rangle \quad (2.2a)$$

or

$$|\Psi_\alpha^{out}\rangle = |\Phi_\alpha\rangle - \frac{1}{H_0 - E_\alpha + i\varepsilon} V |\Psi_\alpha^{out}\rangle \quad (2.2b)$$

where "in" ("out") denotes the boundary condition that in the infinite past (future) the state  $|\Psi^{in}\rangle$  ( $|\Psi^{out}\rangle$ ) behaves like a free state, i.e., the particles in the state do not interact.



Of course bound states, if they exist, cannot be represented in this way; they obey

$$|B\rangle = - \frac{1}{H_0 - E_B} V |B\rangle. \quad (2.3)$$

(2.2a) and (2.2b) are generally known as the Lippmann-Schwinger equations. If we multiply from the left (2.2a), (2.2b) and (2.3) by  $\langle\Phi_\beta|$ , we get what we shall call the 'first' Lippmann-Schwinger equations

$$\langle\Phi_\beta|\Psi_\alpha^{in}\rangle = \langle\Phi_\beta|\Phi_\alpha\rangle - (E_\beta - E_\alpha - i\varepsilon)^{-1} \langle\Phi_\beta|V|\Psi_\alpha^{in}\rangle, \quad (2.4a)$$

$$\langle\Phi_\beta|\Psi_\alpha^{out}\rangle = \langle\Phi_\beta|\Phi_\alpha\rangle - (E_\beta - E_\alpha + i\varepsilon)^{-1} \langle\Phi_\beta|V|\Psi_\alpha^{out}\rangle, \quad (2.4b)$$

and

$$\langle\Phi_\beta|B\rangle = - (E_\beta - E_B)^{-1} \langle\Phi_\beta|V|B\rangle. \quad (2.5)$$

The left hand sides of the last three equations are merely the Fourier transforms of the wave-functions of the respective states if no spin or iso-spin is involved, and if  $\alpha, \beta$  then denotes the momenta of the particles.

Lippmann & Schwinger<sup>(12)</sup> further proved that the S-matrix elements from the state  $\alpha$  to the state  $\beta$  is

$$\begin{aligned} S_{\beta\alpha} &= \langle\Phi_\beta^{out}|\Psi_\alpha^{in}\rangle \\ &= \langle\Phi_\beta|\Phi_\alpha\rangle - 2\pi i \delta(E_\beta - E_\alpha) \langle\Phi_\beta|V|\Psi_\alpha^{in}\rangle, \end{aligned} \quad (2.6a)$$

$$\text{or} \quad S_{\beta\alpha} = \langle\Phi_\beta|\Phi_\alpha\rangle - 2\pi i \delta(E_\beta - E_\alpha) \langle\Phi_\beta^{out}|V|\Phi_\alpha\rangle. \quad (2.6b)$$

We define the "reaction" matrices  $D^{(-)}$  and  $D^{(+)}$  by



$$D_{\beta\alpha}^{(-)} = (\Phi_\beta | V | \Psi_\alpha \text{ in} \rangle, \quad (2.7a)$$

$$D_{\beta\alpha}^{(+)} = \langle \Psi_\beta \text{ out} | V | \Phi_\alpha \rangle \quad (2.7b)$$

for any two states  $\alpha$  and  $\beta$  (on or off the energy shell).

In analogy we define

$$D_{\beta\beta} = (\Phi_\beta | V | \Phi_\beta \rangle. \quad (2.8)$$

Using (2.2a) we get

$$\begin{aligned} D_{\beta\alpha}^{(+)} &= (\Phi_\beta | V | \Phi_\alpha) - \sum_n \frac{(\Phi_\beta | V | \Phi_n)(\Phi_n | V | \Psi_\alpha \text{ in} \rangle}{E_n - E_\alpha - i\varepsilon} \\ &= V_{\beta\alpha} - \sum_n \frac{V_{\beta n} D_{n\alpha}^{(+)}}{E_n - E_\alpha - i\varepsilon} \end{aligned} \quad (2.9a)$$

where the summation over  $n$  extends over a complete set of free states. Similarly from (2.2b) we get

$$D_{\beta\alpha}^{(+)} = V_{\beta\alpha} - \sum_n \frac{D_{\beta n}^{(+)} V_{n\alpha}}{E_n - E_\beta + i\varepsilon}. \quad (2.9b)$$

We shall refer to (2.9a) and (2.9b) as the 'second' Lippmann-Schwinger equations.

On the other hand if we rewrite (2.4a) by making the substitutions

$$|\Psi_\alpha \text{ in} \rangle \rightarrow |\Psi_n \text{ in} \rangle, \quad (\Phi_\beta | \rightarrow (\Phi_\alpha |,$$

take its complex conjugate, multiply throughout by  $(\Phi_\beta | V | \Psi_n \text{ in} \rangle$  and then sum over all the scattering "in" states, we get, on

rearranging, the Low equation for  $D^{(-)}$ :

$$\langle \Phi_\beta | V | \Psi_\alpha^{in} \rangle = \sum_n \left[ \langle \Phi_\beta | V | \Psi_n^{in} \rangle \langle \Psi_n^{in} | \Phi_\alpha \rangle - \frac{\langle \Phi_\beta | V | \Psi_n^{in} \rangle \langle \Psi_n^{in} | V | \Phi_\alpha \rangle}{E_n - E_\alpha - i\epsilon} \right] \quad (2.10a)$$

or

$$D_{\beta\alpha}^{(-)} = V_{\beta\alpha} - \sum_n \frac{D_{\beta n}^{(-)} D_{n\alpha}^{(-)*}}{E_n - E_\alpha - i\epsilon} - \sum_B \frac{D_{\beta B} D_{\alpha B}^*}{E_B - E_\alpha}, \quad (2.10a)$$

with

$$V_{\beta\alpha} = \sum_n \langle \Phi_\beta | V | \Psi_n^{in} \rangle \langle \Psi_n^{in} | \Phi_\alpha \rangle + \sum_B \langle \Phi_\beta | V | B \rangle \langle B | \Phi_\alpha \rangle. \quad (2.11a)$$

The terms involving bound states in (2.10a) cancel out due to (2.5); of course they will not be there if the system has no bound states.

The same sort of manipulations with (2.4b) leads to

$$D_{\beta\alpha}^{(+)} = V_{\beta\alpha} - \sum_n \frac{D_{n\beta}^{(+)*} D_{n\alpha}^{(+)}}{E_n - E_\beta - i\epsilon} - \sum_B \frac{D_{\beta B} D_{\alpha B}^*}{E_B - E_\beta}, \quad (2.10b)$$

the Low equation for  $D^{(+)}$ .

Of course we also have

$$V_{\beta\alpha} = \sum_n \langle \Phi_\beta | \Psi_n^{out} \rangle \langle \Psi_n^{out} | V | \Phi_\alpha \rangle + \sum_B \langle \Phi_\beta | B \rangle \langle B | V | \Phi_\alpha \rangle. \quad (2.11b)$$

At this point we pause to examine equations (2.6a) to (2.10b). The first thing to notice is that  $D_{\beta\alpha}^{(-)}$  and  $D_{\beta\alpha}^{(+)}$  are equal on the energy-shell by virtue of (2.6a) and (2.6b); off the energy-shell, they are most likely different. For if the right hand sides of (2.4a) and (2.4b) are the "Fourier transforms" of the wavefunctions of the "in" and "out" states, then we should expect  $D_{\beta\alpha}^{(-)}$  and  $D_{\beta\alpha}^{(+)}$  to be different.

Before we consider the second point, let us multiply from the

left (2.1) by  $(\Phi_\beta|$ . We get on rearrangement

$$(\Phi_\alpha|V|\Psi_n\rangle^* = -(E_\alpha - E_n)(\Phi_\alpha|\Psi_n\rangle^*, \quad (2.12)$$

in accordance with the general form of a potential between two and an analogous equation for  $D_{\beta\alpha}^{(+)}$ . Substituting (2.12) into (2.10a) shows that if  $E_\alpha$  is small, the coupling of  $D_{\beta\alpha}^{(-)}$  to  $D_{n\alpha}^{(-)}$  and  $D_{\beta n}^{(-)}$  where  $E_n > E_\alpha$  is only apparent so that alteration of the values of the last two will not affect that of the first. Similarly alteration of the values of  $D_{\beta n}^{(+)}$  and  $D_{n\alpha}^{(+)}$  does not affect that of  $D_{\beta\alpha}^{(+)}$  if  $E_n > E_\beta$ . Taken together, these two facts mean that as long as we get the matrix elements between low energies correctly, we shall get low-energy scattering correctly though the potential involves high-energy elements. This is our second point.

The third point to be noticed is that in (2.10a) and (2.10b), the matrix elements of the potential involve only low-energy scattering states if we deal with low energy scattering, whereas (2.9a), (2.9b) necessarily contain matrix elements of the potential involving scattering states of high energy. Since we know the concept of a potential can only hold for low energy phenomena, (the concept of low energy in this context cannot be defined till we come to Chapter 4 where we deal with Q.F.T.), we do not expect to be able to derive in Q.F.T. equations analogous to (2.9a) and (2.9b). We will therefore forget about them in the rest of this chapter and proceed to write out in detail (2.10a) for nucleon-nucleon scattering, with the initial two nucleons specified by  $(\underline{p}, \alpha; \underline{k}, \beta)$  and the final nucleons by  $(\underline{p}', \alpha'; \underline{k}', \beta')$ , the Greek letters denoting the spins and isospins, while the Latin letters the four-momenta.

We shall assume the potential to have the form

$$V(\underline{x}_1, \underline{x}_2; \underline{y}_1, \underline{y}_2) = \delta(\underline{x} - \underline{y}) \delta(\underline{z} - \underline{y}) V(\underline{y}, \nabla_{\underline{y}})$$

in accordance with the general form of a potential between two-nucleons found by Marshak and Okubo<sup>(13)</sup>. This form tells us that the two-nucleons form a closed system, and the non-locality of the interaction is limited to that due to differential operators. Then

$$\begin{aligned} & (\underline{k}', \beta'; \underline{p}', \alpha' | V | \underline{p}, \alpha; \underline{k}, \beta) = \\ &= \frac{\delta(\underline{p} + \underline{k}' - \underline{p} - \underline{k})}{(2\pi)^3} \int d^3x \left\{ e^{-i\underline{z} \cdot (\underline{k}' - \underline{p}')/2} (\beta'; \alpha' | - e^{-i\underline{z} \cdot (\underline{p}' - \underline{k}')/2} (\alpha'; \beta' |) \right\} \times \\ & \quad \times V(\underline{z}, \nabla_{\underline{z}}) e^{i\underline{z} \cdot (\underline{k} - \underline{p})/2} |\alpha, \beta\rangle \\ &= \frac{\delta(\underline{p}' + \underline{k}' - \underline{p} - \underline{k})}{(2\pi)^3} \int d^3x (\beta'; \alpha' | \left\{ e^{-i(\underline{k}' - \underline{p}') \cdot \underline{z}/2} - e^{-i\underline{z} \cdot (\underline{p}' - \underline{k}')/2} P_{\tau} P_{\sigma} \right\} \times \\ & \quad \times V(\underline{z}, \nabla_{\underline{z}}) e^{i\underline{z} \cdot (\underline{k} - \underline{p})/2} |\alpha, \beta\rangle, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} P_{\tau} &= \frac{1}{2} (1 + \underline{\tau}^{(1)} \cdot \underline{\tau}^{(2)}), \\ P_{\sigma} &= \frac{1}{2} (1 + \underline{\sigma}^{(1)} \cdot \underline{\sigma}^{(2)}) \end{aligned} \quad (2.14)$$

are respectively the charge- and the spin-exchange operators. We define  $(\underline{p}' - \underline{k}' | V | \underline{p} - \underline{k})$  by the equation

$$(\underline{k}', \beta'; \underline{p}', \alpha' | V | \underline{p}, \alpha; \underline{k}, \beta) = \delta(\underline{p} + \underline{k}' - \underline{p} - \underline{k}) (\underline{p}' - \underline{k}' | V | \underline{p} - \underline{k}). \quad (2.15)$$

In the rest of this chapter, we shall suppress the spin and iso-spin dependences.

Because the two nucleons form a closed system, the motion of the centre-of-mass must be that of a free particle. This

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\*  $\underline{X}, \underline{Y}$  are the C-M coordinates;  $\underline{x}, \underline{y}$  the relative coordinates.



means that the "Fourier transforms" of the wavefunctions of the states, and therefore through (2.4a) and (2.4b) the matrix elements of  $D^{(+)}$  and  $D^{(-)}$ , must contain a factor  $\delta(\underline{p}_\beta - \underline{p}_\alpha)$ . We therefore define the functions  $\chi^{(+)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k})$ ,  $\chi^{(-)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k})$ ,  $\chi(\underline{k}', \underline{p}'; B)$  and the scattering matrices  $T^{(+)}$ ,  $T^{(-)}$ ,  $T$  by the following equations:

$$\left. \begin{aligned} \delta(\underline{k}' + \underline{p}' - \underline{p} - \underline{k}) T^{(-)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) &= (\underline{k}', \underline{p}' | V | \underline{p}, \underline{k} \text{ in} \rangle, \\ \delta(\underline{k}' + \underline{p}' - \underline{p} - \underline{k}) T^{(+)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) &= \langle \underline{k}', \underline{p}' \text{ out} | V | \underline{p}, \underline{k} \rangle, \\ \delta(\underline{k}' + \underline{p}' - \underline{p} B) T(\underline{k}', \underline{p}'; B) &= (\underline{k}', \underline{p}' | V | B \rangle, \end{aligned} \right\} \quad (2.16)$$

$$\left. \begin{aligned} \delta(\underline{k}' + \underline{p}' - \underline{k} - \underline{p}) \chi^{(-)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) &= (\underline{k}', \underline{p}' | \underline{p}, \underline{k} \text{ in} \rangle, \\ \delta(\underline{k}' + \underline{p}' - \underline{k} - \underline{p}) \chi^{(+)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) &= (\underline{k}', \underline{p}' | \underline{p}, \underline{k} \text{ out} \rangle, \\ \delta(\underline{k}' + \underline{p}' - \underline{p} B) \chi(\underline{k}', \underline{p}'; B) &= (\underline{k}', \underline{p}' | B \rangle. \end{aligned} \right\} \quad (2.17)$$

The Low equation in  $T^{(-)}$  can now be obtained from (2.10a):

$$\begin{aligned} T^{(-)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) &= (\underline{p}' - \underline{k}' | V | \underline{p} - \underline{k}) - \sum_{\underline{p}_n = \underline{p} + \underline{k}} \frac{T^{(-)}(\underline{k}', \underline{p}'; \underline{p}_n) T^{(-)*}(\underline{k}, \underline{p}; \underline{p}_n)}{E_n - E_p - E_k - i\epsilon} \\ &\quad - \sum_{\underline{p}_B = \underline{p} + \underline{k}} \frac{T(\underline{k}', \underline{p}'; B) T^*(\underline{k}, \underline{p}; B)}{E_B - E_p - E_k - i\epsilon} \end{aligned} \quad (2.18)$$

provided  $\underline{p}' + \underline{k}' = \underline{p} + \underline{k}$ .

Notice that in the centre-of-mass system, the functions are essentially the "Fourier transforms" of the wavefunctions and the proviso attached to (2.18) is then redundant.



The equation for  $T^{(+)}(\underline{k}', \underline{p}'; \underline{p}, \underline{k})$  analogous to (2.18) is easily obtained from (2.10b).

Equations (2.10a), (2.10b) do not define  $D^{(-)}$ ,  $D^{(+)}$ , and hence  $T^{(-)}$ ,  $T^{(+)}$ , uniquely. To ensure that their solutions correspond to physical reality, we need the unitarity conditions written symbolically as

$$i(D_{\alpha\beta}^{(-)} - D_{\beta\alpha}^{(-)*}) = (2\pi) \sum_{E_n=E_\alpha} D_{n\alpha}^{(-)*} D_{n\beta}^{(-)}, \quad (2.19a)$$

provided  $E_\alpha = E_\beta$ ;

and 
$$i(D_{\alpha\beta}^{(+)} - D_{\beta\alpha}^{(+)*}) = (2\pi) \sum_{E_n=E_\alpha} D_{n\alpha}^{(+)*} D_{n\beta}^{(+)}, \quad (2.19b)$$

provided  $E_\alpha = E_\beta$ .

The summations extend over a complete set of free states.

$$T(\eta^2, t) = V(t) + \frac{1}{\pi} \int_0^\infty \frac{\lambda T(\eta'^2, t) d\eta'^2}{\eta'^2 - \eta^2 - i\epsilon},$$

where  $t$  is the square of the momentum transfer, and  $\eta$  is the magnitude of the momentum of one particle in the centre-of-mass system. We can write it in a subtracted form:

$$T(\eta^2, t) = T(0, t) + \frac{1}{\pi} \int_0^\infty \frac{\eta'^2}{\eta'^2} \frac{\lambda T(\eta'^2, t) d\eta'^2}{\eta'^2 - \eta^2 - i\epsilon}, \quad (3.1)$$

with

$$V(t) = T(0, t) - \frac{1}{\pi} \int_0^\infty \frac{d\eta'^2}{\eta'^2} \lambda T(\eta'^2, t). \quad (3.2)$$

Suppose we have a charge-exchange potential  $U$ :

$$U(t) = V_1(t) + V_2(t) P_0,$$

### CHAPTER 3

#### METHOD USING DISPERSION RELATIONS

As was pointed out in Chapter 1, the most elegant and satisfactory formulation of this method was put forward by Charap and Fubini<sup>(10),(11)</sup>. In reporting their success here, we shall use a slightly different treatment of the field-theoretical Feynman amplitude, so that in the case of identical particle scattering, a direct appeal to perturbation theory is avoided.

We shall assume in this chapter that nucleons are bosons.

It has been proved by Khuri<sup>(14)</sup> that for a system of two non-identical bosons interacting through a wide class of potentials involving no charge exchange, the T-matrix obeys a one-dimensional dispersion relation:

$$T(\eta^2, t) = V(t) + \frac{1}{\pi} \int_0^\infty \frac{\text{Im} T(\eta'^2, t) d\eta'^2}{\eta'^2 - \eta^2 - i\varepsilon},$$

where  $t$  is the square of the momentum transfer, and  $\eta$  is the magnitude of the momentum of one particle in the centre-of-mass system. We can write it in a subtracted form:

$$T(\eta^2, t) = T(0, t) + \frac{1}{\pi} \int_0^\infty \frac{\eta'^2}{\eta'^2} \frac{\text{Im} T(\eta'^2, t) d\eta'^2}{\eta'^2 - \eta^2 - i\varepsilon}, \quad (3.1)$$

with

$$V(t) = T(0, t) - \frac{1}{\pi} \int_0^\infty \frac{d\eta'^2}{\eta'^2} \text{Im} T(\eta'^2, t). \quad (3.2)$$

Suppose we have a charge-exchange potential  $U$ :

$$U(t) = V_D(t) + V_E(t) P_\tau, \quad (3.6)$$

We turn now to Field Theory.  
 $P_c$  being the charge-exchange operator. If we introduce the  
 projection operators  
 and  $y', k'$  those of the outgoing nucleons. Define  $s, t, u$  by

$$\Lambda_{\pm} = \frac{1}{2} (1 \pm P_c)$$

for isotopic triplet and singlet states, we can write

$$V_{\pm} = V_D \pm V_E,$$

$$T = T_+ \Lambda_+ + T_- \Lambda_-,$$

$$T = T_D + T_E P_c,$$

(3.3)

with

$$T_D = \frac{1}{2} (T_+ + T_-),$$

$$T_E = \frac{1}{2} (T_+ - T_-).$$

(3.4)

$T_+$  and  $T_-$ , being the scattering matrices between isotopic  
 triplet states and singlet states respectively, are not coupled  
 so that they each obey a different one-dimensional dispersion  
 relation of the type (3.1).

In the case of identical-particle scattering, the scattering  
 matrix  $\bar{T}$  is given by

$$\bar{T}(\eta^2, t) = T(\eta^2, t) + T(\eta^2, u), \quad (3.5)$$

where  $G(s, t, u)$  is the Feynman amplitude. For this section we  
 where  $u$  is the square of the momentum-transfer, alternative to  
 $t$ . If the potential involves charge exchange, we have

$$\bar{T} = \bar{T}_+ \Lambda_+ + \bar{T}_- \Lambda_-, \quad (3.6)$$

with

$$\bar{T}_{\pm} = T_{\pm}(\eta^2, t) \pm T_{\pm}(\eta^2, u). \quad (3.7)$$

We turn now to Field Theory.

Let  $p, k$  denote the four-momenta of the in-going nucleons, and  $p', k'$  those of the out-going nucleons. Define  $s, t, u$  by

$$s = -(p+k)^2 = -(p'+k')^2 \geq 4m^2 \quad \text{for physical scattering,}$$

$$t = -(p'-p)^2 = -(k-k')^2 < 0 \quad \text{for physical scattering,}$$

$$u = -(p'-k)^2 = -(k-p')^2 < 0 \quad \text{for physical scattering,}$$

Because we are dealing with physical scattering processes so that we have conservation of energy and momentum, there is a relationship between  $s, t, u$ :

$$s + t + u = 4m^2$$

In the centre-of-mass system,  $t$  and  $u$  will be identical to the  $t$  and  $u$  in (3.5).  $m$  is the mass of a nucleon.

#### (I) Non-identical nucleons:- \*

We will first assume that the nucleons are coupled to neutral mesons only. The field-theoretical S-matrix can be written, in the C-M system as

$$S(p', k'; p, k) = -(2\pi)^4 \delta^4(p'+k'-p-k) \frac{m^2}{p^0} G(s, t, u) + \delta(p'-p) \delta(k'-k), \quad (3.8)$$

where  $G(s, t, u)$  is the Feynman amplitude. For this section we suppress the iso-spin labels.

Mandelstam<sup>(15)</sup> suggested on the basis of Perturbation Theory that  $G(s, t, u)$  may obey a double dispersion relation of the form

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\*  $p$  and  $k$  denote different kinds of particles. We do not allow the transition  $p \rightarrow k + \pi$ .



$$\begin{aligned}
 G(s, t, u) = & \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{\rho_1(t')}{t' - t} + \frac{1}{\pi} \int_{u_0}^{\infty} du' \frac{\rho_2(u')}{u' - u} + \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho_3(s')}{s' - s} + \\
 & + \frac{1}{\pi^2} \int_{t_0}^{\infty} dt' \int_{u_0}^{\infty} du' \frac{\rho_{12}(t', u')}{(t' - t)(u' - u)} + \frac{1}{\pi^2} \int_{t_0}^{\infty} dt' \int_{s_0}^{\infty} ds' \frac{\rho_{13}(t', s')}{(t' - t)(s' - s)} + \\
 & + \frac{1}{\pi^2} \int_{u_0}^{\infty} du' \int_{s_0}^{\infty} ds' \frac{\rho_{23}(u', s')}{(u' - u)(s' - s)}, \quad (3.9)
 \end{aligned}$$

where  $s_0$ ,  $t_0$ ,  $u_0$  are the lowest masses of the physical states which have the same quantum numbers as the channels in which  $s$ ,  $t$ ,  $u$  are respectively the energies squared. In the present case

$$\begin{aligned}
 s_0 &= 4m^2, \\
 t_0 &= \mu^2, \quad \mu \text{ being the pion mass,} \\
 u_0 &= 4m^2.
 \end{aligned}$$

All the weight functions are real, and in particular  $\rho_{12}$ ,  $\rho_{13}$ ,  $\rho_{23}$  are non-vanishing only in some unphysical domains, typified by the shaded area in Fig. 1 for  $\rho_{12}$ . Henceforth we shall assume that (3.9) has a validity independent of Perturbation Theory.

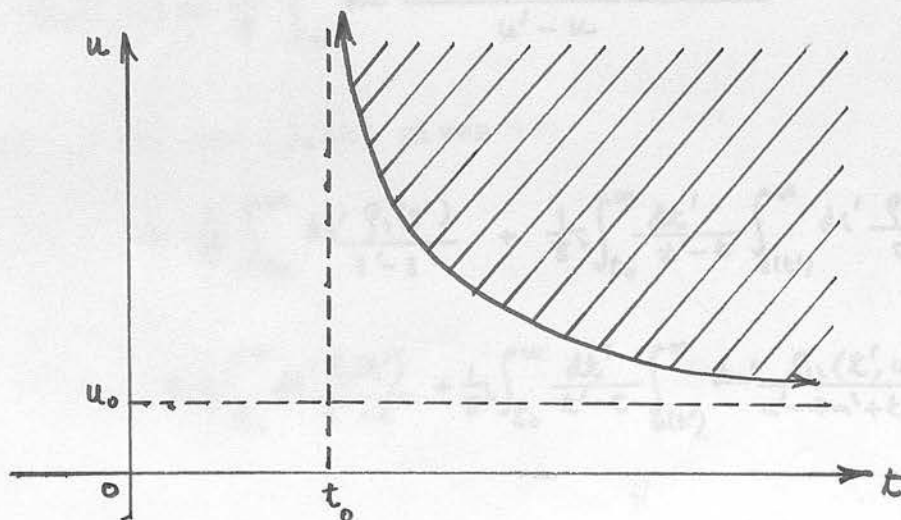


Fig. 1.



(3.9) can be converted into a one-dimensional dispersion relation:

$$G(s, t, u) = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho_3(s')}{s' - s} + \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{A_1(s, t', 4m^2 - t' - s)}{t' - t} + \frac{1}{\pi} \int_{u_0}^{\infty} du' \frac{A_2(s, 4m^2 - u' - s, u')}{u' - u}, \quad (3.10)$$

where  $A_1(s, t, u) = \rho_1(t) + \frac{1}{\pi} \int_{u(t)}^{\infty} du' \frac{\rho_{12}(t, u')}{u' - u} + \frac{1}{\pi} \int_{s(t)}^{\infty} ds' \frac{\rho_{13}(t, s')}{s' - s}, \quad (3.11)$

and  $A_2(s, t, u) = \rho_2(u) + \frac{1}{\pi} \int_{t(u)}^{\infty} dt' \frac{\rho_{12}(t', u)}{t' - t} + \frac{1}{\pi} \int_{s(u)}^{\infty} ds' \frac{\rho_{23}(u, s')}{s' - s} \quad (3.12)$

are the absorptive parts for the channels,  $t, u$  respectively,  $t(u)$  being the point  $u$  on the bounding curve of the domain in which  $\rho_{12}$  does not vanish, etc. We now write

$$G(s, t, u) = G_1(s, t) + G_2(s, u), \quad (3.13)$$

with  $G_1(s, t) = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho_3(s')}{s' - s} + \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{A_1(s, t', 4m^2 - t' - s)}{t' - t}, \quad (3.14)$

and  $G_2(s, u) = \frac{1}{\pi} \int_{u_0}^{\infty} du' \frac{A_2(s, 4m^2 - s - u', u')}{u' - u} \quad (3.15)$

Combining (3.11) and (3.14) gives

$$G_1(s, t) = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho_3(s')}{s' - s} + \frac{1}{\pi^2} \int_{t_0}^{\infty} \frac{dt'}{t' - t} \int_{s(t')}^{\infty} ds' \frac{\rho_{13}(t', s')}{s' - s} + \frac{1}{\pi} \int_{t_0}^{\infty} \frac{dt'}{t' - t} \rho_1(t') + \frac{1}{\pi^2} \int_{t_0}^{\infty} \frac{dt'}{t' - t} \int_{u(t')}^{\infty} du' \frac{\rho_{12}(t', u')}{u' - 4m^2 + t' + s}, \quad (3.16a)$$

Thus (3.16b) can be written as

$$= \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho_3(s')}{s'-s} + \frac{1}{\pi^2} \int_{s_0}^{\infty} \frac{ds'}{s'-s} \int_{t(s')}^{\infty} dt' \frac{\rho_{12}(t', s')}{t'-t} +$$

$$+ \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{\rho_1(t')}{t'-t} + \frac{1}{\pi^2} \int_{t_0+u_0}^{\infty} \frac{dz'}{z'+s-4m^2} \int_{t(z')}^{\infty} dt' \frac{\mathbb{H}(t', z')}{t'-t}, \quad (3.16b)$$

where  $x' = t' + u'$ ,  $\mathbb{H}(t', x') = \rho_{12}(t', x'-t')$  which is non-vanishing in the shaded area in Fig. II.

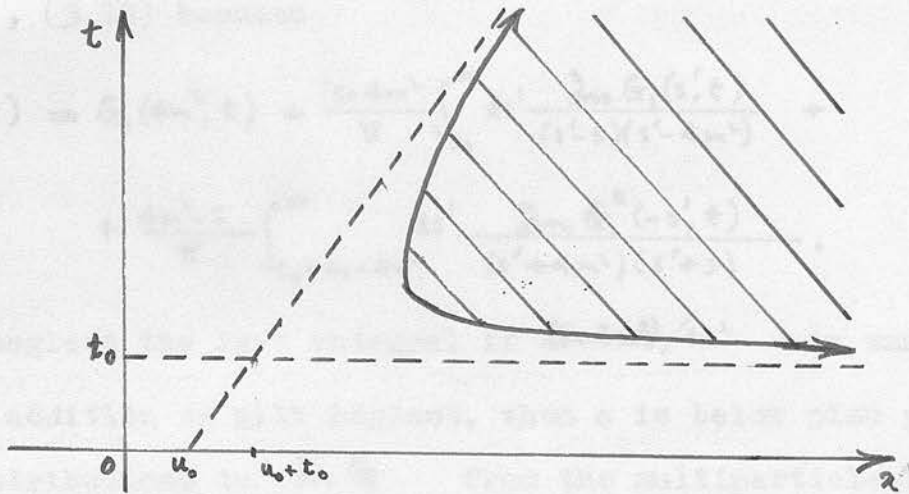


Fig. II.

When we have physical nucleon-nucleon scattering, i.e., when  $s \geq s_0$  and  $t < 0$ , we get from (3.16b)

$$\text{Im } G_1(s, t) = \rho_3(s) + \frac{1}{\pi} \int_{t(s)}^{\infty} dt' \frac{\rho_{12}(t', s)}{t'-t}. \quad (3.17)$$

If we have unphysical nucleon-nucleon scattering so that  $s < 0$ ,  $t < 0$ , we get, again from (3.16b),

$$\text{Im } G_1(s, t) = \frac{1}{\pi} \int_{t(s)}^{\infty} dt' \frac{\mathbb{H}(t', 4m^2-s)}{t'-t} \equiv \text{Im } G_1^a(s, t). \quad (3.18)$$

Thus (3.16b) can be written as

$$G_1(s, t) = \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{f_1(t')}{t' - t} + \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } G_1(s', t)}{s' - s} + \frac{1}{\pi} \int_{t_0 + u_0 - 4m^2}^{\infty} ds' \frac{\text{Im } G_1^a(-s', t)}{s' + s}, \quad (3.19)$$

which is just a one-dimensional dispersion relation for  $G_1(s, t)$  with  $t$  kept fixed. If we make a subtraction in  $s$  at the point  $s = 4m^2$ , (3.19) becomes

$$G_1(s, t) = G_1(4m^2, t) + \frac{s - 4m^2}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } G_1(s', t)}{(s' - s)(s' - 4m^2)} + \frac{4m^2 - s}{\pi} \int_{t_0 + u_0 - 4m^2}^{\infty} ds' \frac{\text{Im } G_1^a(-s', t)}{(s' + 4m^2)(s' + s)}. \quad (3.22)$$

We can now neglect the last integral if  $(s - 4m^2)/4m^2$  is small enough. In addition we will neglect, when  $s$  is below pion production, contributions to  $\text{Im } G_1$  from the multiparticle intermediate states because they do not contribute till  $s'$  is above the threshold of pion production and therefore not appreciably. Thus the last equation is reduced to

$$G_1(s, t) = G_1(4m^2, t) + \frac{s - 4m^2}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } G_1^{el}(s', t)}{(s' - s)(s' - 4m^2)}. \quad (3.20)$$

Similarly  $G_2(s, u)$  can be transformed by means of (3.12) into

$$G_2(s, u) = \frac{1}{\pi} \int_{u_0}^{\infty} \frac{du' f_2(u')}{u' - u} + \frac{1}{\pi} \int_{s_0}^{\infty} \frac{ds'}{s' - s} \int_{u(s')}^{\infty} \frac{du' f_2(u', s')}{u' - u} + \frac{1}{\pi} \int_{u_0}^{\infty} du' \frac{1}{u' - u} \int_{t(u')}^{\infty} dt' \frac{f_2(t', u')}{t' - 4m^2 + u' + s}. \quad (3.21)$$

Since  $u_0 = 4m^2$  and  $u < 0$  for physical nucleon-nucleon scattering, it is easily seen that each term in (3.21) is small compared with the corresponding term in (3.16b) if  $(s-4m^2)/4m^2$  is small. Thus for low  $s$ , we have

$$\frac{m^2}{p^2} G(s, t, u) \doteq G_1(s, t).$$

Substituting  $s' = 4(m^2 + \eta'^2)$ ,  $s = 4(m^2 + \eta^2)$ , we can now compare (3.20) with (3.1) for low energies provided we identify

$$T(0, t) \text{ with } G_1(4m^2, t) \text{ so that}$$

$$V(t) = G_1(4m^2, t) - \frac{1}{\pi} \int_0^\infty d\eta'^2 \frac{\text{Im} T(\eta'^2, t)}{\eta'^2}. \quad (3.22)$$

Comparison with the potential obtained by Charap and Fubini<sup>(10)</sup> shows that it agrees exactly with (3.22), though we have thrown away a little more of  $G(s, t, u)$  in our procedure.

The introduction of charged mesons into this model does not give rise to any difficulty. We merely have

$$\begin{aligned} G(s, t, u) &= G_A(s, t, u) + \xi'' \cdot \xi''' G_B(s, t, u) \\ &= \Lambda_+ G_+(s, t, u) + \Lambda_- G_-(s, t, u), \end{aligned} \quad (3.23)$$

where all the  $G$ 's are invariant functions. We assume  $G_\pm$  each obeys a Mandelstam representation, and analyse each in exactly the same way as before to get

$$V_\pm(t) = G_{\pm 1}(4m^2, t) - \frac{1}{\pi} \int_0^\infty d\eta'^2 \frac{\text{Im} T_\pm(\eta'^2, t)}{\eta'^2}. \quad (3.24)$$



(II) Identical nucleons:-

In this case (3.8) has the additional term

$$+ \delta(\phi' - \frac{1}{2}) \delta(\frac{1}{2} - \phi).$$

We again first assume that the nucleons are coupled to neutral mesons only. Clearly we can forget about the isospin indices for the moment. (3.10), (3.11) and (3.12) remain unchanged in form but, since the nucleons form a boson system, we must have

$$G(s, t, u) = G(s, u, t)$$

which means

$$\left. \begin{aligned} \rho_1(t) &= \rho_2(t), \\ \rho_{12}(t, u) &= \rho_{21}(u, t), \\ \rho_{23}(t, s) &= \rho_{13}(t, s). \end{aligned} \right\} \quad (3.25)$$

and

The lower limits remain the same except one:

$$u_0 = \mu^2.$$

We now define  $G'_1(s, t)$  and  $G'_2(s, u)$  by

$$G(s, t, u) = G'_1(s, t) + G'_2(s, u), \quad (3.26)$$

with 
$$G'_1(s, t) = \frac{1}{2\pi} \int_0^\infty ds' \frac{\rho_3(s')}{s' - s} + \frac{1}{\pi} \int_{t_0}^\infty dt' \frac{A_1(s, t', 4m^2 - t' - s)}{t' - t}, \quad (3.27)$$

$$G'_2(s, u) = \frac{1}{2\pi} \int_0^\infty ds' \frac{\rho_3(s')}{s' - s} + \frac{1}{\pi} \int_{u_0}^\infty du' \frac{A_2(s, 4m^2 - u' - s, u')}{u' - u}. \quad (3.28)$$

From (3.11), (3.12), (3.25), (3.27) and (3.28) we see that  $G'_1$  and  $G'_2$  have exactly the same functional form, and that the break-up of  $G(s, t, u)$  as shown in (3.26) is exactly the same as in (3.5). Thus we need only analyse  $G'_1$  or  $G'_2$ , which can

be done in exactly the same way as was done to  $G_1$ , provided  $(t' + u)$  in the last integral in (16a) is bigger than or equal to  $m^2$ , i.e., provided  $[t' + u(t')] \geq m^2$ . If this last inequality does not hold, it means our way of defining the potential breaks down. The potential is again of the form given by (3.22).

If charged mesons are admitted we again split  $G(s, t, u)$  into two parts as in (3.23), each part obeying a double representation of the form (3.9) and a single representation of the form (3.10). The symmetry properties of  $G^+$  and  $G^-$  are expressed by the relations:

$$\left. \begin{aligned} \rho_1^+(t) &= \rho_2^+(t), \\ \rho_{12}^+(t, u) &= \rho_{12}^+(u, t), \\ \rho_{13}^+(t, s) &= \rho_{23}^+(t, s); \end{aligned} \right\} \quad (3.29)$$

and

$$\left. \begin{aligned} \rho_3^-(s) &= 0, \\ \rho_1^-(t) &= -\rho_2^-(t), \\ \rho_{12}^-(t, u) &= -\rho_{12}^-(u, t), \\ \rho_{23}^-(t, s) &= -\rho_{13}^-(t, s). \end{aligned} \right\} \quad (3.30)$$

(If a subtraction is needed in  $t$ , clearly one will be needed in  $u$  as well. In that event we shall make the subtractions at the same value of  $t$  and  $u$  to preserve the symmetry properties of the weight functions.)

We define  $G_1^\pm(s, t)$ ,  $G_2^\pm(s, u)$  by

$$G_\pm(s, t, u) = G_1^\pm(s, t) \pm G_2^\pm(s, u), \quad (3.31)$$

with  $G_1^+$ ,  $G_2^+$  having representations of the forms of (3.27) and (3.28) respectively, while  $G_1^-$  and  $G_2^-$  have no integrals containing  $\rho_3^-(s)$ . We can therefore find in the same manner two potentials  $V_+$  and  $V_-$  and get

$$V(t) = \Lambda_+ V_+(t) + V_-(t) \Lambda_-.$$

It is now clear that this handling of the Feynman amplitude in Q.F.T. can be readily extended to the realistic case of fermion-fermion scattering, where we have ten invariant functions of  $s$ ,  $t$ ,  $u$ . Each of these functions is either symmetric or anti-symmetric with respect to the interchange  $t \leftrightarrow u$  by virtue of Pauli's Exclusion Principle. The only reason why we cannot proceed in the same way to find a potential is that we do not know about the analytic properties of the scattering matrix  $T$  for a potential with general spin dependence.

Let us return to our boson model.

The Mandelstam representation alone will probably not give a unique solution to  $G(s, t, u)$ . To choose the physical one we need the unitarity condition which in the case of Field Theory will be of the same form as that of potential-scattering, i.e., of the form of (2.19), if the energy is below pion production; and as the energy becomes large enough, will have additional terms corresponding to intermediate states with more particles than two nucleons and baryon number two.

Up to now in this chapter, we only considered low energy scattering so that certain terms in the field-theoretical amplitude by the present method and by the naive substitution of the field-

amplitude can be neglected, and the formal comparison can then be made with the T-matrix element of a potential theory. The philosophy of this method is that we will obtain a potential theory which gives the low energy scattering amplitude reasonably correct if we assume that the approximation procedure outlined above can be carried out even for high energies, and that the unitarity condition is of the same form as (2.19). Obviously the amplitude at high energies given by such a potential theory will be entirely different from that given by Field Theory. The second assumption cuts out completely the effect on  $\text{Im } G$  of particle-creation in the intermediate states. This, rather than the factor  $m^2/p^2$ , is the important difference between  $\text{Im } G(s, t, u)$  and  $\text{Im } T(q^2, t)$  as calculated from such a potential theory for high energies; i.e., the latter is the elastic part of the former. Hence we think that the assertion by Charap and Fubini<sup>(10)</sup> that there is a difference between their method and that of Goldberger, Nambu and Oehme<sup>(16)</sup> is ill-founded.

For convenience, we have assumed so far in this chapter that the deuteron does not exist. Its inclusion, however, presents no additional difficulty and leads essentially to a further term in the potential as given by (3.22), namely,  $(-1)$  times

$$C(4m^2, t)$$

which is the additional contribution from the deuteron as an intermediate state to  $G(4m^2, t, u)$ . In their paper<sup>(10)</sup>, Charap and Fubini left this term out deliberately, and hence their assertion that to fourth order in the coupling constant the potentials given by the present method and by the naive substitution of the field-



theoretical T matrix elements into the "second" Lippmann-Schwinger equation (2.9) are equal, is not valid if they admit the deuteron.

Before closing this chapter, we want to make one more remark. It is often surmised (see Goldberger et al.<sup>(16)</sup> for example) that if one writes a one-dimensional dispersion relation for in s, keeping t fixed, then the integral arising from the left-hand cut, the pole terms, and possibly contributions from in-elastic processes associated with the right-hand cut, will together give the potential for the elastic process in the s-channel. We want to emphasize that this is true only for the scattering of non-identical particles. In the case of identical particles, an important part of the integral from the left-hand cut is not included in the potential, as a comparison of

$$V(t) + V(u)$$

with the last mentioned dispersion relation will show.

In spite of the elegance and success of this method, it cannot be at present generalized to the realistic case. What is more, the processes of approximation are justified by surmises only, e.g., that the weight functions in (3.9) due to the various intermediate states are of the correct relative magnitudes. We therefore attempt to find a more satisfactory method in this next chapter.

Note: Since the completion of this thesis, Chew and Frautschi<sup>(33)</sup> proposed to define the "generalized potentials" thus:

$$V_1(s, t) = G_1(s, t) - \int \frac{ds'}{\pi} \frac{\rho_1(s', t)}{s' - s},$$

and

$$V_2(s, u) = G_2(s, u) - \int \frac{ds'}{\pi} \frac{\rho_2(s', u)}{s' - s},$$

which obviously are approximately equal to our potentials (3.22) and its equivalent in  $u$  when the energy is within the range of validity of the last two. Their "potentials" are highly energy-dependent and when used to calculate the scattering amplitude, will produce inelastic cuts in the complex energy plane. This second feature does not tally with potential scattering. So they went on to define a potential theory, using Froissart's method to eliminate the inelastic cuts in each partial amplitude. However, in this reduction the elastic phase shift becomes modified, even in the low energy region, to an extent dependent on the rate of increase in importance of the inelastic processes. Apart from the case of the S-wave amplitude, there is no simple connection between the potential and the jump across the left-hand cut. It is therefore doubtful whether their "potential theory" is anything more than nominal; certainly it will be difficult to extract from their theory a potential as is understood in Quantum Mechanics.

annihilation operators. The order of appearance of the labels in a state vector corresponds to that of the operators acting on the vacuum. The amplitude can be rewritten in the following way (17):

$$\begin{aligned}
 & \langle k', p' \text{ out} | p, k \text{ in} \rangle \\
 &= \lim_{x^0 \rightarrow \infty} \int_{\Sigma} d^3x \sqrt{\frac{2\pi}{V^0}} \frac{1}{(2\pi)^{3/2}} e^{-i p' x} \langle k' | \bar{u}(x) \not{p} u(x) | p, k \text{ in} \rangle \\
 &= \sqrt{\frac{2\pi}{V^0}} \frac{1}{(2\pi)^{3/2}} \left[ \int_{-\infty}^{\infty} d^3x \frac{2}{x^0} \{ e^{-i p' x} \langle k' | \bar{u}(x) \not{p} u(x) | p, k \text{ in} \rangle \} + \right. \\
 & \quad \left. + \lim_{x^0 \rightarrow \infty} \int_{\Sigma} d^3x e^{-i p' x} \langle k' | \bar{u}(x) \not{p} u(x) | p, k \text{ in} \rangle \right], \quad (4.1)
 \end{aligned}$$

# CHAPTER 4

## T-MATRIX ELEMENTS, THE LOW EQUATION, AND THE POTENTIAL IN Q.F.T.

In this chapter we base ourselves on the axiomatic Q.F.T. first formulated by Lehmann, Symanzik, and Zimmermann<sup>(17)</sup>. In this theory the transition amplitude from a state of two nucleons with four-momenta  $p, k$  (actually we use these to denote spins and iso-spins as well when they are called for) to a state of two nucleons with four-momenta  $p', k'$  is

$$\langle k', p' \text{ out} | p, k \text{ in} \rangle,$$

where

$$|p, k \text{ in} \rangle = a_{\text{in}}^{\dagger}(p) a_{\text{in}}^{\dagger}(k) |0\rangle,$$

and

$$\langle k', p' \text{ out} | = \langle 0 | a_{\text{out}}(k') a_{\text{out}}(p'),$$

$|0\rangle$  being the "dressed" vacuum, and  $a$ 's the renormalized annihilation operators. The order of appearance of the labels in a state vector corresponds to that of the operators acting on the vacuum. The amplitude can be rewritten in the following way<sup>(17)</sup>:

$$\begin{aligned} & \langle k', p' \text{ out} | p, k \text{ in} \rangle \\ &= \lim_{x^0 \rightarrow \infty} \int_{x^0} d^3z \sqrt{\frac{m}{p^0}} \frac{1}{(2\pi)^{3/2}} e^{-ip' \cdot z} \langle k' | \bar{u}(p') \beta \psi(z) | p, k \text{ in} \rangle \\ &= \sqrt{\frac{m}{p^0}} \frac{1}{(2\pi)^{3/2}} \left[ \int_{-\infty}^{\infty} d^4z \frac{\partial}{\partial z^0} \{ e^{-ip' \cdot z} \langle k' | \bar{u}(p') \beta \psi(z) | p, k \text{ in} \rangle \} + \right. \\ & \quad \left. + \lim_{x^0 \rightarrow -\infty} \int_{x^0} d^3z e^{-ip' \cdot z} \langle k' | \bar{u}(p') \beta \psi(z) | p, k \text{ in} \rangle \right], \quad (4.1) \end{aligned}$$

where  $u(p')$  is a spinor for a nucleon with four-momenta, etc. denoted by  $p'$ ,  $p' \cdot x$  is the scalar product and the metric is  $(-1, 1, 1, 1)$ . The second integral can be cast into the form

$$\langle k' | a_{in}(p') | p, k in \rangle = \langle p' | p \rangle \langle k' | k \rangle - \langle p' | k \rangle \langle k' | p \rangle$$

so that

$$\begin{aligned} & \langle k', p' out | p, k in \rangle \\ &= \sqrt{\frac{m}{p'^0}} \frac{i}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^4x \bar{u}(p') \left\{ \gamma^0 \frac{\partial}{\partial x^0} + \gamma^i \frac{\partial}{\partial x^i} - \gamma^i \frac{\partial}{\partial x^i} + m - m \right\} x \\ & \quad \times \{ e^{-ip' \cdot x} \langle k' | \psi(x) | p, k in \rangle \} + \langle p' | p \rangle \langle k' | k \rangle - \langle p' | k \rangle \langle k' | p \rangle. \end{aligned}$$

Use of Dirac's equation for  $\bar{u}(p') e^{-ip' \cdot x}$  and integration by parts give

$$\begin{aligned} & \langle k', p' out | p, k in \rangle \\ &= \langle p' | p \rangle \langle k' | k \rangle - \langle p' | k \rangle \langle k' | p \rangle + \\ & \quad + \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m}{p'^0}} \int_{-\infty}^{\infty} d^4x e^{-ip' \cdot x} \bar{u}(p') \langle k' | (\gamma \cdot \partial + m) \psi(x) | p, k in \rangle \\ &= \langle p' | p \rangle \langle k' | k \rangle - \langle p' | k \rangle \langle k' | p \rangle + \\ & \quad + \frac{i}{(2\pi)^{3/2}} (2\pi)^4 \delta^4(p' + k' - p - k) \sqrt{\frac{m}{p'^0}} \langle k' | \bar{u}(p') f(0) | p, k in \rangle \end{aligned} \quad (4.2)$$

where we have used

$$\psi(x) = e^{-iP \cdot x} \psi(0) e^{+iP \cdot x},$$

$$f(x) = (\gamma \cdot \partial + m) \psi(x),$$

and

$$\bar{f}(x) = \bar{\psi}(x) (\gamma \cdot \overleftarrow{\partial} + m).$$



$P$  is the energy-momentum operator of the system. The first of these equations is strictly not correct as it stands, but is to be taken between two states.

Alternatively, we can "take out"  $k'$  instead of  $p'$  and get

$$\begin{aligned} & \langle k', p' \text{ out} | p, k \text{ in} \rangle \\ &= \langle p' | p \rangle \langle k' | k \rangle - \langle p' | k \rangle \langle k' | p \rangle - \\ & \quad - \frac{i}{(2\pi)^{3/2}} (2\pi)^4 \delta^4(p' + k' - p - k) \sqrt{\frac{m}{k_0}} \langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle. \end{aligned} \quad (4.3)$$

In analogy with potential theory we now define our  $D^{(-)}$  matrix in Q.F.T. as follows:

$$\begin{aligned} D^{(-)}(k', p'; p, k) &= \frac{(2\pi)^{3/2}}{2} \delta^4(p' + k' - p - k) \left[ \sqrt{\frac{m}{k_0}} \langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle - \right. \\ & \quad \left. - \sqrt{\frac{m}{p_0}} \langle k' | \bar{u}(p') f(0) | p, k \text{ in} \rangle \right]. \end{aligned} \quad (4.4a)$$

Notice there is no conservation of energy. The reason for the averaging here is that neither term on the right-hand side is antisymmetric under the interchange  $p' \leftrightarrow k'$  whereas such is the case of  $D^{(-)}$  as defined by (2.7a). (In the case of boson-boson scattering, this same averaging automatically makes  $D^{(-)}$  in (4.4a) symmetric under the interchange of  $p'$  and  $k'$ , in agreement with  $D^{(-)}$  given by (2.7a)). As will be shown presently in this chapter, this averaging is crucial to the success of our theory.

In the same way, by "taking out" the particles in the "in"

state, we will define  $D^{(*)}$  as

$$D^{(*)}(k', p'; p, k) = \frac{(2\pi)^{3/2}}{2} \delta(p' + k' - p - k) \left[ \sqrt{\frac{m}{k_0}} \langle k', p' \text{ out} | \bar{f}(0) u(k) | p \rangle - \sqrt{\frac{m}{p_0}} \langle k', p' \text{ out} | \bar{f}(0) u(p) | k \rangle \right]. \quad (4.4b)$$

For a bound state  $|B\rangle$  we define  $D$  as

$$D(k', p'; B) = \frac{(2\pi)^{3/2}}{2} \delta(p' + k' - p_B) \left[ \sqrt{\frac{m}{k_0}} \langle p' | \bar{u}(k') f(0) | B \rangle - \sqrt{\frac{m}{p_0}} \langle k' | \bar{u}(p') f(0) | B \rangle \right]. \quad (4.5)$$

Now

$$\begin{aligned} & \langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle \\ &= \lim_{x \rightarrow 0} \bar{u}(k') (\gamma \cdot \partial + m) \langle p' | \psi(x) | p, k \text{ in} \rangle \\ &= \bar{u}(k') \{ i \gamma \cdot (p + k - p') + m \} \langle p' | \psi(0) | p, k \text{ in} \rangle \\ &= (p_0 + k_0 - p'_0 - k'_0) \langle p' | \bar{u}(k') \beta \psi(0) | p, k \text{ in} \rangle. \end{aligned} \quad (4.6)$$

In the last step we have made use of the Dirac equation as well as the conservation of three-momentum. We can do the same sort of manipulations to the other matrix elements in (4.4a), (4.4b) and (4.5). Thus we can define in analogy with (2.12) the "Fourier transforms" of the wavefunctions for a state with in-going particles  $p, k$ , a state with out-going particles  $p, k$ , and a bound state respectively as

$$X^{(*)}(k', p'; p, k) = \frac{(2\pi)^{3/2}}{2} \delta(k' + p' - k - p) \left[ \langle k' | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle \sqrt{\frac{m}{p_0}} - (k' \longleftrightarrow p') \right], \quad (4.7a)$$

$$X^{(+)}(k', p'; p, k) = \frac{(2\pi)^{3/2}}{2} \delta(k' + p' - p - k) \left[ \sqrt{\frac{m}{p^0}} \langle k' | \bar{u}(p') \beta \psi(0) | p, k_{out} \rangle - (k' \leftrightarrow p') \right], \quad (4.7b)$$

and

$$X(k', p'; B) = \frac{(2\pi)^{3/2}}{2} \delta(k' + p' - p_B) \left[ \sqrt{\frac{m}{p^0}} \langle k' | \bar{u}(p') \beta \psi(0) | B \rangle - (k' \leftrightarrow p') \right]. \quad (4.8)$$

Also in analogy with (2.16) and (2.17) we can define the T-matrices and  $\chi$ -functions by dropping the delta-functions in (4.4), (4.5), (4.7) and (4.8).

At this stage we are in a position to find the analogues of (2.4a) and (2.4b); that of (2.5) is of course trivially obtained from the definition of the various quantities involved. Let us examine then the expression

$$\begin{aligned} & \sqrt{\frac{m}{p^0}} \langle k' | \bar{u}(p') \beta \psi(0) | p, k_{in} \rangle \\ &= \sqrt{\frac{m^2}{p^0 k^0}} \frac{1}{(2\pi)^{3/2}} \lim_{x^0 \rightarrow \infty} \int_{x^0} d^3x e^{-ik'x} \langle 0 | \bar{u}(k') \beta \psi(x) \bar{u}(p') \beta \psi(0) | p, k_{in} \rangle \\ &= \sqrt{\frac{m^2}{p^0 k^0}} \frac{1}{(2\pi)^{3/2}} \left[ \int d^4x \frac{\partial}{\partial x^0} \{ e^{-ik'x} \langle 0 | T(\bar{u}(k') \beta \psi(x) \bar{u}(p') \beta \psi(0)) | p, k_{in} \rangle \} \right. \\ & \quad \left. - \lim_{x^0 \rightarrow -\infty} \int_{x^0} d^3x e^{-ik'x} \langle 0 | \bar{u}(p') \beta \psi(0) \bar{u}(k') \beta \psi(0) | p, k_{in} \rangle \right] \\ &= \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m^2}{p^0 k^0}} \int d^4x e^{-ik'x} \left[ \langle 0 | T\{\bar{u}(k') f(x) \bar{u}(p') \beta \psi(0)\} | p, k_{in} \rangle \right. \\ & \quad \left. + \delta(x^0) \langle 0 | \{\bar{u}(k') \gamma^0 \psi(x), \bar{u}(p') \beta \psi(0)\} | p, k_{in} \rangle \right] \\ & \quad - \sqrt{\frac{m}{p^0}} \langle k' | p \rangle \langle 0 | \bar{u}(p') \beta \psi(0) | k \rangle + \sqrt{\frac{m}{p^0}} \langle k' | k \rangle \langle 0 | \bar{u}(p') \beta \psi(0) | p \rangle. \end{aligned}$$

The second integral in the last line vanishes by virtue of microscopic causality, and the last two terms give respectively,

if  $p' + k' - p - k = 0$  is taken into account,

$$(m/p^0)^{1/2} \langle k' | p \rangle \bar{u}(p') \beta u(k) = \langle k' | p \rangle \langle p' | k \rangle,$$

$$(m/p^0)^{1/2} \langle k' | k \rangle \bar{u}(p') \beta u(p) = \langle k' | k \rangle \langle p' | p \rangle.$$

The symbol  $T$  is Wick's chronological operator defined by

$$T \{ f(x) \bar{\psi}(x') \} = \theta(x-x') \{ f(x), \bar{\psi}(x') \} - \bar{\psi}(x') f(x),$$

and

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \frac{e^{i\lambda x}}{\lambda - i\varepsilon}.$$

Insertion of a complete set of states (consisting of all the "in" states and the bound states) between the two operators in the first term then leads to

$$\begin{aligned} & (m/p^0)^{1/2} \langle k' | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle \\ &= \langle k' | k \rangle \langle p' | p \rangle - \langle k' | p \rangle \langle p' | k \rangle + \sqrt{\frac{m^2}{p^0 k^0}} \frac{1}{(2\pi)^{3/2}} \times \\ & \times \left[ \sum_{p_n = k'} (2\pi)^3 \frac{\langle 0 | \bar{u}(k') f(0) | n \rangle \langle n | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle}{p_n^0 - k'^0 - i\varepsilon} + \right. \\ & \left. + \sum_{p_n = p + k - k'} (2\pi)^3 \frac{\langle 0 | \bar{u}(p') \beta \psi(0) | n \rangle \langle n | \bar{u}(k') f(0) | p, k \text{ in} \rangle}{-p_n^0 - k'^0 + p^0 + k^0 + i\varepsilon} \right]. \end{aligned}$$

The intermediate states in both sums must have baryon number equal to one. In the first sum states with only one nucleon but no other particles do not contribute. Hence using manipulations similar to those that led to (4.6) we get

$$\begin{aligned} & (m/p^0)^{1/2} \langle k' | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle \\ &= \langle k' | k \rangle \langle p' | p \rangle - \langle k' | p \rangle \langle p' | k \rangle + \sqrt{\frac{m^2}{p^0 k^0}} (2\pi)^{3/2} \times \\ & \times \left[ - \sum'_{p_n = k'} \langle 0 | \bar{u}(k') \beta \psi(0) | n \rangle \langle n | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle + \right. \\ & + \sum'_{p_n = p'} \frac{\langle 0 | \bar{u}(p') \beta \psi(0) | n \rangle \langle n | \bar{u}(k') \beta \psi(0) | p, k \text{ in} \rangle (p^0 + k^0 - p_n^0 - k'^0)}{-p_n^0 - k'^0 + p^0 + k^0 + i\varepsilon} + \\ & \left. + \frac{p^0}{m} \frac{\langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle}{-p^0 - k'^0 + p^0 + k^0 + i\varepsilon} \right], \end{aligned} \quad (4.9)$$



provided  $p' + k' - p - k = 0$ .

The sign  $\sum'$  denotes the fact that the intermediate states do not include the single-nucleon states the contribution from which appear in the last term.

Similarly we get an expression for

$$\sqrt{\frac{m}{k^0}} \langle p' | \bar{u}(k') \beta \psi(0) | p, k \text{ in} \rangle,$$

which is essentially the same as (4.9), but with  $p'$  and  $k'$  interchanged.

Substitution of these two expressions into (4.7a) then shows that if

$$p^0 + k^0 - k'^0 < m + \mu,$$

i.e., if  $p^0 + k^0$  is below the pion-production threshold, the contributions from intermediate states with more than one particle cancel one another exactly, and that we get

$$\begin{aligned} X^{(-)}(k', p'; p, k) &= \langle k' | k \rangle \langle p' | p \rangle - \langle k' | p \rangle \langle p' | k \rangle - \\ &- \frac{D^{(-)}(k', p'; p, k)}{p^0 + k^0 - p^0 - k^0 - i\varepsilon}, \end{aligned} \quad (4.10a)$$

provided  $p^0 + k^0 < 2m + \mu$ .

In exactly the same way we get

$$\begin{aligned} X^{(+)}(k', p'; p, k) &= \langle k' | k \rangle \langle p' | p \rangle - \langle p' | k \rangle \langle k' | p \rangle - \\ &- \frac{D^{(+)*}(k, p; p', k')}{p^0 + k^0 - p^0 - k^0 + i\varepsilon}, \end{aligned} \quad (4.10b)$$

provided  $p^0 + k^0 < 2m + \mu$ .

Thus we get our "first" Lippmann-Schwinger equations entirely within the framework of Q.F.T. The procedure that led to (2.10a) can be repeated now and produces the Low equation:

$$D^{(-)}(k', p'; p, k) = V(k', p'; p, k) - \sum_B \frac{D(k', p'; B) D^*(k, p, B)}{p_B^0 - p^0 - k^0} - \sum_{\substack{\text{2-nucleon} \\ \text{states}}} \frac{D^{(-)}(k', p'; p_n, k_n) D^{(-)*}(k, p; p_n, k_n)}{p_n^0 + k_n^0 - p^0 - k^0 - i\epsilon}, \quad (4.11a)$$

provided  $p^0 + k^0 < 2m + \mu$ , and

$$V(k', p'; p, k) = \sum_{\substack{\text{2-nucleon} \\ \text{states}}} D^{(-)}(k', p'; p_n, k_n) X^{(-)*}(k, p; p_n, k_n) - \sum_B D(k', p'; B) X^*(k, p; B).$$

Although (4.10a) is only valid if the in-coming state has energy below  $2m + \mu$ , we nevertheless used it as if it were true for all physical energies in deriving (4.11a). This is legitimate as long as  $p^0 + k^0 < 2m + \mu$  because in this case the high-energy states actually contribute nothing to (4.11a) as was mentioned in Chapter 2.

Similarly we get from (4.10b)

$$D^{(+)}(k', p'; p, k) = U(k', p'; p, k) - \sum_B \frac{D(k', p'; B) D^+(k, p; B)}{p_B^0 - p'^0 - k'^0} - \sum_{\substack{\text{2-nucleon} \\ \text{states}}} \frac{D^{(+)*}(k_n, p_n; p', k') D^{(+)}(k_n, p_n; p, k)}{k_n^0 + p_n^0 - p'^0 - k'^0 - i\epsilon}, \quad (4.11b)$$

provided  $p'^0 + k'^0 < 2m + \mu$ , and

$$\mu(k', p'; p, k) = \sum_{\substack{\text{2-nucleon} \\ \text{states}}} X^{(+)}(k', p'; p_n, k_n) D^{(+)}(k_n, p_n; p, k) - \\ - \sum_B X(k', p'; B) D^*(k, p; B).$$

Although  $V$  in (4.11a) and  $\mu$  in (4.11b) have the same structure as their counterparts in (2.10a) and (2.10b) which are equal, we labelled them differently because they are apparently different quantities in Q.F.T. We shall come back to this point soon.

If  $p^0 + k^0$  [ $p'^0 + k'^0$ ] is below the pion-production threshold, the unitarity condition in terms of  $D^{(-)}$  [ $D^{(+)}$ ] will have exactly the same form as (2.19a) [(2.19b)]. Above this threshold, modification will be the same as discussed in Chapter 3.

We now assert that if we assume equations (4.10a) to (4.11b), and the unitary in the form of (2.19) to hold for all physical energies for the "in" or "out" states involved, we will have a potential theory for a system of two nucleons.

The potential theory thus obtained will give us exactly the same scattering amplitudes as axiomatic Q.F.T. if the "in" or "out" states involved have energies below pion production. The reason is, as pointed out before, that the coupling, by (4.11), of the amplitude with the scattering state at low energy to those with the scattering states at high energies is fictitious, and thus the distortion of the latter implied in our method does not affect the former. This fact actually gives us complete freedom to assign values to the scattering amplitudes involving "in" or "out" states at energy above

the pion-production threshold - a not inconvenient freedom when we come to calculate the potential.

It is well-known (e.g. see Ekstein<sup>(19)</sup>) that a very wide variety of potentials can lead to the same S-matrix, i.e., the same T-matrix elements that are on the energy shell. Since only the scattering-matrix elements on the energy-shell have physical meaning, we can define those off the energy-shell quite arbitrarily, getting thereby a wider variety of potentials. Viewing the matter from this angle, we indeed have no reason to suppose that  $U$  and  $V$  in (4.11) should be equal, products as they are of two different extrapolations. However, we saw in Chapter 2 that, given a potential, two ways of extrapolating the D-matrices off the energy-shell in terms of the potential, taken between two states led to equations (2.4), (2.5) and (2.10). In Q.F.T. we did the same thing to the D-matrices but in terms of the current operator  $f(\mathbf{0})$  taken between two states, and obtained a set of analogous equations. The strong parallel between the two theories rather induces us to suspect that, at least when  $p^0 + k^0$  and  $p'^0 + k'^0$  are below the pion-production threshold,  $U$  and  $V$  in (4.11) are probably equal, at worst approximately, provided we choose consistently in both cases the values for the D-matrix elements involving high-energy scattering states. One consistent choice is probably to put them all equal to zero, which in configuration space means that the potential will be flat when the particles are within a definite distance of each other. Unfortunately, we have no means of coming to a decision in the present state of Q.F.T.



In our claim that we have derived a potential theory from Q.F.T. we have slurred over one point. To justify our definition of the "Fourier transform" of the wave function of an "in" state, e.g., we must be able to prove that (4.7a) defines a set of functions which obey the orthonormal condition, at least when the energies of the "in" states are below pion-production. We are unable to do this. However it is easy to get back to the wave function from its "Fourier transform"; we merely multiply (4.7a) by

$$\frac{1}{\sqrt{2}} \left\{ e^{-i(\underline{p}' \cdot \underline{z}_1 + \underline{k}' \cdot \underline{z}_2)} |\alpha', \beta'\rangle - e^{-i(\underline{p}' \cdot \underline{z}_2 + \underline{k}' \cdot \underline{z}_1)} |\beta', \alpha'\rangle \right\},$$

then integrate over  $\underline{p}'$  and  $\underline{k}'$ , and sum over  $\alpha', \beta'$ .  $|\alpha', \beta'\rangle$  is the direct product of two two-dimensional Pauli spinors and two two-dimensional iso-spinors, completely labelled by  $\alpha'$  and  $\beta'$ . Since (4.7a) is essentially a scalar function of the quantum numbers specifying the "in" state and the other variables will be "integrated" out, the wave function so obtained is susceptible to the usual interpretations - a decisive advantage over the "wave function" suggested by Gell-Mann and Low<sup>(4)</sup>. Apart from trivial kinematic factors, (4.8) is the same as the definition used by Blankenbecler and Cook<sup>(20)</sup> except that they did not antisymmetrise with respect to  $\underline{p}'$  and  $\underline{k}'$ .

It should be pointed out that if we can prove the orthonormality of our wavefunctions, two results follow. First, if we change in (4.10a)  $\underline{k}'$  to  $\underline{k}_n$ ,  $\underline{p}'$  to  $\underline{p}_n$ , multiply through by  $V(\underline{k}', \underline{p}'; \underline{k}_n, \underline{p}_n)$  and finally "sum" over  $\underline{p}_n$  and  $\underline{k}_n$ , we get

$$\sum_n V(k', p'; p_n, k_n) X^{(-)}(k_n, p_n; p, k) = V(k', p'; p, k) - \sum_n \frac{V(k', p'; p_n, k_n) D^{(-)}(k_n, p_n; p, k)}{k_n^0 + p_n^0 - p^0 - k^0 - i\varepsilon}. \quad (4.12)$$

Application of the definition of  $V$  and the orthonormality of the  $X$ 's then reduces the left-hand side to  $D^{(-)}(k', p'; p, k)$ . That is, (4.12) is essentially the "second" Lippmann-Schwinger equation for  $D^{(-)}$ . The one for  $D^{(+)}$  can be similarly obtained. Secondly, a similar sort of manipulation with (4.10a) leads, together with the orthonormality of our wavefunctions, to

$$\frac{D^{(+)*}(k, p; p', k') - D^{(-)}(k', p'; p, k)}{p^0 + k^0 - p'^0 - k'^0 - i\varepsilon} + \sum_n \frac{D^{(-)}(k', p'; p_n, k_n) D^{(+)*}(k, p; p_n, k_n)}{(p^0 + k^0 - p_n^0 - k_n^0 - i\varepsilon)(p'^0 + k'^0 - p_n^0 - k_n^0 + i\varepsilon)} + \sum_B \frac{D(k', p'; B) D^*(k, p; B)}{(k^0 + p^0 - 2m + B)(k'^0 + p'^0 - 2m + B)} = 0. \quad (4.13)$$

This has been used by Braun.<sup>(21)</sup>

From (4.11a), we easily get that

$$- [V(k', p'; p, k) - V^*(k, p; p', k')] / (p^0 + k^0 - p^0 - k^0 - i\varepsilon)$$

is equal to the left-hand side of (4.13), and thus the potential  $V$  is Hermitian.<sup>\*\*</sup> Similarly for  $U$ . Thus our theory would be complete.

The derivation of (4.11) may appear to be a bit contrived. We will therefore show the connection between (4.11) and the expressions we normally obtain in proving dispersion relations. For this purpose we examine again  $\langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle$ .

---

<sup>\*\*</sup> This consequence has been observed by Haag.<sup>(26)</sup>

$$\begin{aligned}
 & \langle k' | \bar{u}(p') f(0) | p, k \rangle \\
 &= \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \lim_{x^0 \rightarrow -\infty} \int_{x^0} d^3x e^{ip \cdot x} \langle k' | \bar{u}(p') f(0) \bar{\psi}(x) \beta u(p) | k \rangle \\
 &= \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \left[ - \int_{-\infty}^{\infty} d^4x \frac{\partial}{\partial x^0} \left\{ e^{ip \cdot x} \langle k' | T(\bar{u}(p') f(0) \bar{\psi}(x) \beta u(p)) | k \rangle \right\} - \right. \\
 & \quad \left. - \lim_{x^0 \rightarrow \infty} \int_{x^0} d^3x e^{ip \cdot x} \langle k' | \bar{\psi}(x) \beta u(p) \bar{u}(p') f(0) | k \rangle \right]. \quad (4.14)
 \end{aligned}$$

The last term can be rewritten as

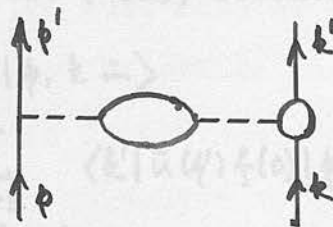
$$\begin{aligned}
 & \langle k' | a_{out}^\dagger(p) \bar{u}(p') f(0) | k \rangle = \langle k' | p \rangle \langle 0 | \bar{u}(p') f(0) | k \rangle = 0. \\
 & \text{The routine manipulation of the rest of (4.14) will then give} \\
 & \langle k' | \bar{u}(p') f(0) | p, k \rangle \\
 &= \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \left[ \sum_{p_n = p + k} \frac{\langle k' | \bar{u}(p') f(0) | n \rangle \langle n | \bar{f}(0) u(p) | k \rangle (2\pi)^3}{p_n^0 - p^0 - k^0 - i\varepsilon} + \right. \\
 & \quad + \sum_{p_n = p' - p} \frac{\langle k' | \bar{f}(0) u(p) | n \rangle \langle n | \bar{u}(p') f(0) | k \rangle (2\pi)^3}{-p_n^0 - p^0 + p'^0 + i\varepsilon} + \\
 & \quad \left. + \int_{-\infty}^{\infty} d^4x e^{ip \cdot x} \langle k' | \delta(x^0) \{ \bar{u}(p') f(0), \psi^\dagger(x) u(p) \} | k \rangle \right], \quad (4.15)
 \end{aligned}$$

where the summation is over a complete set of states. This is the usual expression. Since  $p$  and  $p'$  are four-momenta of physical nucleons  $(p_n^0 + p^0 - p'^0)$  never vanishes in the second "sum" in (4.15), and we can therefore use a relation analogous to (4.6) to rewrite this "sum" as

$$\begin{aligned}
 & - (2\pi)^3 \sum_{p_n = p' - p} \langle k' | \psi^\dagger(0) u(p) | n \rangle \langle n | \bar{u}(p') f(0) | k \rangle \\
 &= - \int_{-\infty}^{\infty} d^4x \langle k' | \delta(x^0) \psi^\dagger(x) u(p) \bar{u}(p') f(0) | k \rangle
 \end{aligned}$$

which therefore cancels exactly one of the terms in the equal

time anticommutator in (4.15). Now this last mentioned anticommutator can be simplified if we adopt a specific interaction between the nucleon and pion fields (e.g. pseudoscalar coupling) and after integration can be represented graphically thus:



It is well known that due to the "bare" vertex between  $p$  and  $p'$ , the matrix element corresponding to this graph contains an unrenormalized coupling constant, and is therefore divergent. To compensate for this a divergent contribution must be contained in either the first or the second sum in (4.15). This means that the contribution to (4.15) by a particular intermediate state cannot be the same as that to a dispersion integral, for otherwise the dispersion integral will be divergent. If the second sum in (4.15) is divergent, it clearly does not matter to us since we are going to drop it and its counterpart in the equal-time integral. If the first sum is divergent again it does not matter as we immediately proceed to demonstrate. By inserting a complete set of states into it, we can rewrite the remaining part of the equal-time integral in (4.15) thus

$$\sum_{n=p+k} (2\pi)^3 \langle k' | \bar{u}(p') f(0) | n \rangle \langle n | \bar{\psi}(0) \beta u(p) | k \rangle.$$

Comparing this last expression with the first sum in (4.15) we see that if we use the technique that led to (4.6) on

$$\langle n | \bar{f}(0) u(p) | k \rangle, \text{ only the intermediate states with}$$



energy around  $p^0 + k^0$  actually contribute to (4.15); the right-hand side of (4.15) is therefore finite for any particular value of  $p^0 + k^0$ . In particular if we keep  $p^0 + k^0$  below  $2m + \mu$ , we can retain only two-nucleon and the deuteron intermediate states and (4.15) becomes

$$\begin{aligned} & \langle k | \bar{u}(p) f(0) | p, k \text{ in} \rangle \\ &= (2\pi)^{3/2} \sqrt{\frac{m}{p^0}} \cdot \sum_{\substack{p_n = p+k \\ \text{(2-nucleon +} \\ \text{bound states)}}} \langle k' | \bar{u}(p') f(0) | p_n \rangle \left\{ \frac{\langle p_n | \bar{f}(0) u(p) | k \rangle}{p_n^0 - p^0 - k^0 - i\epsilon} + \right. \\ & \quad \left. + \langle p_n | \bar{\psi}(0) \beta u(p) | k \rangle \right\}. \end{aligned}$$

We can handle  $\langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle$  in the same way. Substitution into (4.7a) then leads to (4.11a). (4.11b) can be similarly obtained.

Although we have developed our formalism in nucleon-nucleon scattering, it can obviously be extended to cover the scattering of two particles of any description. If a boson is involved and we "take it out", then instead of

$$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \langle k' | \bar{u}(p') f(0) | p, k \text{ in} \rangle,$$

we get

$$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{1}{2p^0}} \langle k' | j(0) | p, k \text{ in} \rangle,$$

where

$$j(x) = (\partial^\mu \partial_\mu + \mu^2) \phi(x),$$

$\mu$  and  $\phi(x)$  being the mass and the field operators for the boson respectively. Instead of

$$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \langle k' | \bar{u}(p') \beta \psi(0) | p, k \text{ in} \rangle,$$

we get

$$-\frac{1}{(2\pi)^4} \sqrt{\frac{1}{2p^0}} (p^0 - k^0 + k^0 + p^0) \langle k' | \phi(0) | p, k \rangle.$$

In each specific case, the energy range within which the Low equations and the "first" Lippmann-Schwinger equations are valid is limited by the rest mass of the lightest particle that can be produced by the collision under discussion. Hence in our discussion of the nucleon-nucleon problem, we have actually confined ourselves to strong interactions, for otherwise particles lighter than the pion can be produced and our equations (4.10) to (4.11) will be exact in a much smaller energy region; in particular we have excluded electromagnetic interaction which, if introduced, would reduce the energy region to zero. This last fact is rather gratifying because electromagnetic interaction can be treated satisfactorily only by a relativistic theory, even in Classical Physics.

Having come so far with our formalism, the most obvious question we like to ask is: Can we extend it to scattering of more than two particles? This extension is not trivial. Let us consider the  $D^{(n)}$ -matrix. For two-particle scattering, the bras in (4.4a) are single-particle states, i.e., they are steady states. If we define  $D^{(n)}$  for multiparticle scattering again by comparing the S-matrix elements, the bras will be "out" states which are not steady. This fact alters the picture so much that if we again define  $X^{(n)}$  by an equation analogous to (4.7a), we cannot establish the "first" Lippmann-Schwinger equation for the simplest case of multiparticle

scattering, i.e., three-particle scattering, even if we severely restrict the energies of the "in" and "out" states simultaneously. Now as a system has more particles in it, a smaller average kinetic energy for the particles will add up to enough energy for the production of a given particle, and hence as a general rule, a potential theory will have a smaller domain of validity in the sense that it describes the system adequately only when the latter has less motion. But we certainly do not expect the theory to break down so quickly as to do so in the three-particle case. Of course in the last-mentioned case, the "first" Lippmann-Schwinger equation may still hold approximately, but this can be proved either way only when we can calculate the various matrix elements involved in a satisfactory way.

Since we can only hope that the result is qualitatively correct at best, we will make an expansion in powers of  $\hbar^2/m^2$  and  $\hbar^2/\mu^2$  etc., and neglect terms of order  $\hbar^2/m^2$  etc. smaller than the main term, but only after we have made sure that this kind of approximation is justified.

We start off by considering  $V$  defined in (4.11a).

If we write

$$V(\mathbf{k}, \mathbf{p}, \mathbf{q}, \mathbf{k}) = \delta(\mathbf{0}) V(\mathbf{k}; \mathbf{k}), \quad (5.1)$$

one of the terms of  $V(\mathbf{k}; \mathbf{k})$  is

$$= - \frac{1}{2} \sum_{\mathbf{p}, \mathbf{q}} \frac{1}{\hbar^2} \langle \mathbf{k} | a(\mathbf{p}) f(\mathbf{0}) | \mathbf{n} \rangle \langle \mathbf{n} | \bar{a}(\mathbf{0}) \bar{a}(\mathbf{q}) | \mathbf{k} \rangle$$

$$= - \frac{1}{2} \frac{1}{\hbar^2} \int d\mathbf{p}_1 d\mathbf{p}_2 \delta(\mathbf{p}_1 + \mathbf{p}_2) \langle \mathbf{k} | a(\mathbf{p}) f(\mathbf{0}) | \mathbf{p}_1, \mathbf{p}_2 \rangle \times \langle \mathbf{p}_1, \mathbf{p}_2 | \bar{a}(\mathbf{0}) \bar{a}(\mathbf{q}) | \mathbf{k} \rangle, \quad (5.2)$$

## CHAPTER 5

### CALCULATION OF THE POTENTIAL

In this chapter we will work exclusively in the centre-of-momentum system so that we need only work with the T-matrices and  $\mathcal{K}$ -functions. In the absence of a better method, we shall use perturbation theory, under the assumption that with the momentum of each nucleon kept below  $\mu$ , it is sufficiently accurate to calculate to fourth order in the coupling constant  $g_0$ , pseudoscalar coupling being assumed:

$$H(x) = i g_0 \bar{\Psi}(x) \gamma_5 \epsilon \cdot \phi(x) \Psi(x).$$

Since perturbation theory is not reliable, we can only hope that the result is qualitatively correct at best. We will make expansion in powers of  $\underline{k}^2/m^2$  and  $\underline{k}'^2/m^2$  etc., and neglect terms of order  $\underline{k}^2/m^2$  etc. smaller than the main term, but only after we have made sure that this kind of approximation is justified.

We start off by considering  $V$  defined in (4.11a).

If we write

$$V(\underline{k}', \underline{p}'; \underline{p}, \underline{k}) = \delta(0) V(\underline{k}'; \underline{k}), \quad (5.1)$$

one of the terms of  $V(\underline{k}'; \underline{k})$  is  $-\langle \underline{k}', \underline{p}'; \underline{p}, \underline{k} \rangle$

$$\equiv - \sum_{\substack{\underline{p}_1 = 0 \\ \text{(2-nucleon} \\ \text{states)}}} (2\pi)^3 \sqrt{\frac{m^2}{k'^0 k^0}} \langle \underline{k}' | \bar{u}(\underline{p}') f(0) | n \rangle \langle n | \bar{\Psi}(0) \beta u(\underline{p}) | \underline{k} \rangle$$

$$= - \frac{(2\pi)^3}{2} \sqrt{\frac{m^2}{k'^0 k^0}} \int d^3 p_1 d^3 p_2 \delta(\underline{p}_1 + \underline{p}_2) \langle \underline{k}' | \bar{u}(\underline{p}') f(0) | \underline{p}_2, \underline{p}_1, in \rangle \times \\ \times \langle \underline{p}_1, \underline{p}_2, in | \bar{\Psi}(0) \beta u(\underline{p}) | \underline{k} \rangle, \quad (5.2)$$



where the integrations actually include summation over spin and isospin variables. The factor  $\frac{1}{2}$  is there because  $|p_1, p_2 \text{ in}\rangle$  and  $|p_2, p_1 \text{ in}\rangle$  are essentially the same so that by including them both in our integral we have counted the same state twice.

We now go over to the interaction picture which coincides with the Heisenberg picture at  $t = 0$ . The complex conjugate of the second matrix element in the integrand of (5.2) then transforms as follows:

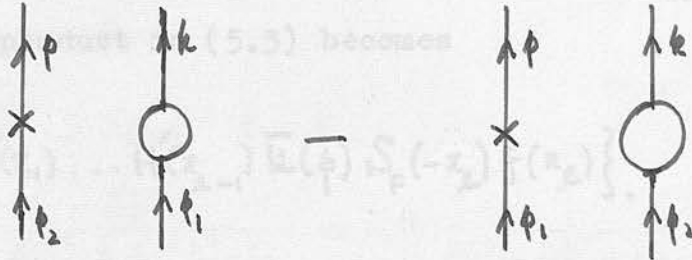
$$\begin{aligned} & \langle k | \bar{u}(p) \beta \psi(0) | p_2, p_1 \text{ in} \rangle \\ &= \langle k | U^\dagger(0, \infty) \bar{u}(p) \beta \psi^{\text{int}}(0) U(0, -\infty) | p_2, p_1 \rangle \\ &= \langle k | T \{ U(\infty, -\infty) \bar{u}(p) \beta \psi^{\text{int}}(0) \} | p_2, p_1 \rangle \\ &= \sum_{l=0}^{\infty} \frac{(-i)^l}{l!} \int d^4 x_1 \dots d^4 x_l \langle k | T \{ H'(x_1) \dots H'(x_l) \bar{u}(p) \beta \psi(0) \} | p_2, p_1 \rangle \end{aligned} \quad (5.2)$$

where  $T$  is Wick's chronological operator, the bras and kets with rounded brackets are state-vectors in interaction picture, and  $U(\infty, -\infty)$  is the operator that transforms an interaction-picture state at  $t = -\infty$  to that at  $t = +\infty$ . In the second and third lines of (5.3) every operator is an operator in the interaction picture; we drop the label "int" for brevity.

We can divide the terms in (5.3) into two classes. To the first class belong all those obtained when we use  $\psi(0)$  to annihilate one of the particles  $p_2, p_1$  so that the remaining one must go through energy and momentum conserving vertices to become

(5.5)

particle  $k$ . The sum of these terms can be represented graphically thus:



where the "blobs" represent all possible energy-momentum conserving processes and a cross indicates that at such a "vertex" we merely have a factor  $\beta$  instead of the various factors according to the usual rules and only conservation of three-momentum. The contribution to (5.3) from this class then is

$$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{k^0}} \left[ \bar{u}(p_2) \beta u(p) \langle p_1 | k \rangle - \bar{u}(p_1) \beta u(p) \langle p_2 | k \rangle \right],$$

and to (5.2) is

$$-\frac{1}{2} (2\pi)^{3/2} \sqrt{\frac{m^2}{k'^0 k^0}} \sqrt{\frac{m}{k^0}} \int d^3 p_2 \delta(k + p_2) \left[ \langle k' | \bar{u}(p') f(0) | p_2, k \text{ in} \rangle \times \right. \\ \left. \times \bar{u}(p_2) \beta u(p) - (k \longleftrightarrow p_2) \right]$$

$$= - (2\pi)^{3/2} \sqrt{\frac{m}{k'^0}} \langle k' | \bar{u}(p') f(0) | p, k \text{ in} \rangle. \quad (5.4)$$

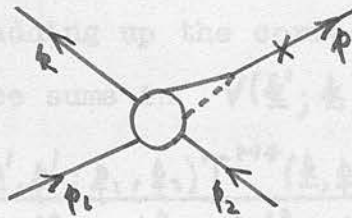
If we take into consideration the corresponding terms from the other three expressions that make up  $V(k'; k)$ , the contribution of this class to  $V(k'; k)$  is

$$T(k', p'; p, k). \quad (5.5)$$

To the second class of contributions to (5.3) belong those which are obtained by contracting  $\bar{\psi}(0)$  with the operator  $\bar{\psi}(x)$  in one of the interaction Hamiltonian density, say  $H'(x_e)$ , so that the time-ordered product in (5.3) becomes

$$-T\{H'(x_1)\dots H'(x_{e-1})\bar{u}(p)S_F(-x_e)f(x_e)\}. \quad (5.6)$$

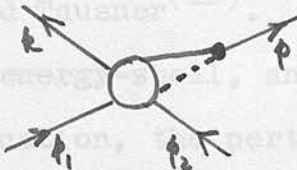
The sum of terms in this class can be represented graphically as



The internal nucleon line next to the arrow  $p$  corresponds to  $S_F$  in (5.6) and represents a nucleon with four-momentum

$$p_1 + p_2 - k = (2p_1^0 - k^0, -\underline{k}).$$

If we compare this graph with



which represents clearly

$$\langle k' | \bar{u}(p) f(0) | p_2, p_1 \text{ in} \rangle,$$

we see the two differ only by the factor

$$-i \frac{(2\pi)^4}{(2\pi)^4} \frac{i[\gamma \cdot (p_1 + p_2 - k) - m]}{(p_1 + p_2 - k)^2 + m^2 - i\varepsilon} = -i(2\pi)^4 S_F(p_1 + p_2 - k).$$

which comes immediately after  $\bar{u}(p)\beta$ . In the last graph, a black dot represents a vertex which carries a coupling constant and a delta-function conserving three-momentum only. Using the Dirac equation we easily get

$$\begin{aligned} & -i \bar{u}(p) \beta S_F(p_1 + p_2 - k) \\ &= \bar{u}(p) \frac{-2p_1^0}{-4p_1^0 + 4p_1^0 k^0 - i\varepsilon} = \bar{u}(p) \frac{1}{2p_1^0 - 2k^0 + i\varepsilon}. \end{aligned}$$

Thus putting into (5.2) the contributions to (5.3) from this second class, and adding up the corresponding contributions from the other three sums in  $V(k'; k)$ , we get

$$\sum_{p_1 + p_2 = 0} \frac{T^{(1)}(k', p'; p_1, p_2) T^{(1)*}(k, p; p_1, p_2)}{p_1^0 + p_2^0 - 2k^0 - i\varepsilon}.$$

This means first that in perturbation theory  $T^{(1)}$  obeys the Low equation, and secondly that we can get our potential  $V(k'; k)$  by calculating the  $T^{(1)}$ -matrix elements in perturbation theory and put them into (2.18). This is not the same as the naive approach of Charap and Tausner<sup>(22)</sup>. There they did not define the T-matrix off the energy-shell, and they used the "second" Lippmann-Schwinger equation, the perturbation expansion of which is probably not valid if there is a bound state<sup>(23)</sup>.

We now proceed to calculate  $V(k'; k)$  to fourth order in  $g$ . In perturbation theory the lowest order contribution

to We shall put our result in matrix form:

$$V(k'; k) = \sum_{i=1}^6 V_i(k', k, t) Q_i - (k \leftrightarrow k') P_0 P_2,$$



where

$$Q_1 = 1,$$

$$Q_2 = \frac{1}{2}(\sigma^{(1)} \cdot \underline{k} \sigma^{(1)} \cdot \underline{k}' + \sigma^{(2)} \cdot \underline{k} \sigma^{(2)} \cdot \underline{k}') = \underline{u}^2 - \underline{t}^2 - i(\sigma^{(1)} + \sigma^{(2)}) \cdot \underline{u} \wedge \underline{t},$$

$$Q_3 = \sigma^{(1)} \cdot \underline{u} \wedge \underline{t} \sigma^{(2)} \cdot \underline{u} \wedge \underline{t},$$

$$Q_4 = \sigma^{(1)} \cdot \underline{t} \sigma^{(2)} \cdot \underline{t},$$

$$Q_5 = \sigma^{(1)} \cdot \sigma^{(2)}, \quad Q_6 = \sigma^{(1)} \cdot \underline{u} \sigma^{(2)} \cdot \underline{u},$$

$$\underline{t} = \frac{1}{2}(\underline{k}' - \underline{k}), \quad \underline{u} = \frac{1}{2}(\underline{k}' + \underline{k}). \quad (5.7)$$

We shall also need the following explicit representations of  $u(k)$ , etc.:

$$u(k) = \frac{-i \underline{k} \cdot \gamma + m}{\{2m(k^0 + m)\}^{1/2}},$$

and

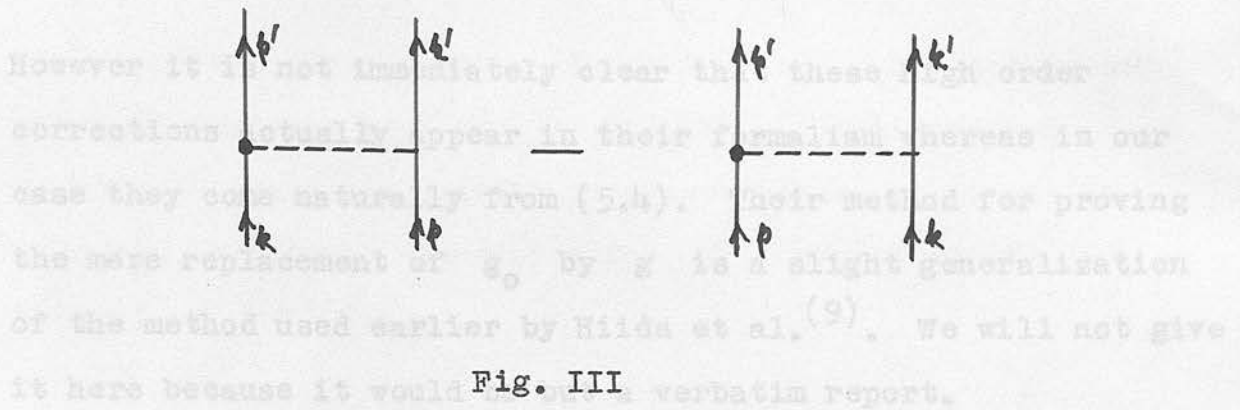
$$\gamma^0 = -i\beta = -i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$\gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i=1, 2, 3.$$

(i) The one-meson-exchange potential  $V^{(2)}$  :-

In perturbation theory the lowest order contribution to (5.4) is, graphically

and concluded that if  $|\underline{k}|$  and  $|\underline{k}'|$  are small enough, i.e.,  $\ll \Lambda$ , these corrections merely give (5.7) with the unrenormalized coupling constant  $g_0$  replaced by the renormalized one  $g$ .

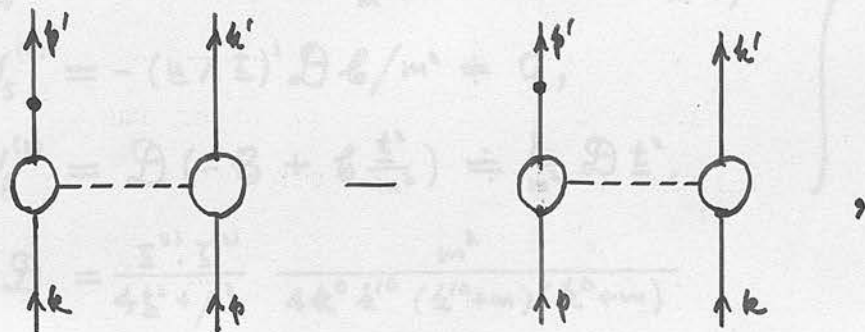


and explicitly the same contribution as (5.7) comes from the

$$\frac{g_0^2}{(2\pi)^3} \sum_{j=1}^3 \left[ \frac{\bar{u}(k') \gamma_5 \tau_j u(k) \bar{u}(p') \gamma_5 \tau_j u(p)}{(k-k')^2 + \mu^2} - (k \leftrightarrow p) \right], \quad (5.7)$$

where we have made the approximation  $(k'^0 - k^0)^2 \doteq 0$ .

This approximated expression has been obtained by every past author in the field when it is on the energy-shell; off the energy-shell, to our knowledge, only Hoshizaki and Machida<sup>(24)</sup> succeeded, using method 2 outlined in Chapter 1. They actually went on in the same paper to discuss the effects of higher order corrections to the graphs in Fig. III, namely,



and concluded that if  $|k|$  and  $|k'|$  are small enough, i.e.,  $\ll 3\mu$ , these corrections merely give (5.7) with the unrenormalized coupling constant  $g_0$  replaced by the renormalized one  $g$ .

However it is not immediately clear that these high order corrections actually appear in their formalism whereas in our case they come naturally from (5.4). Their method for proving the mere replacement of  $g_0$  by  $g$  is a slight generalization of the method used earlier by Hiida et al.<sup>(9)</sup>. We will not give it here because it would be but a verbatim report.

Essentially the same contribution as (5.7) comes from the other matrix element in (5.5). Thus

$$V^{(2)} = \frac{g^2}{(2\pi)^3} \left[ \sum_{j=1}^3 \frac{\bar{u}(k') \gamma_5 \tau_j u(k) \bar{u}(p') \gamma_5 \tau_j u(p)}{(k' - k)^2 + \mu^2 - (k'^0 - k^0)^2} - (p \leftrightarrow k) \right]$$

$$\doteq \frac{g^2}{(2\pi)^3} \left[ \sum_{i=1}^6 V_i^{(2)} Q_i - (t \leftrightarrow u) P_\tau P_\sigma \right], \quad (5.8)$$

with

$$\left. \begin{aligned} V_1^{(2)} &= V_2^{(2)} = 0, \\ V_3^{(2)} &= \mathcal{D} b \left( \frac{1}{m} \right)^2 \doteq \frac{8}{m^2} \mathcal{D}, \\ V_4^{(2)} &= \mathcal{D} (-A + b \frac{u^2}{m^2}) \doteq \mathcal{D} (-16 - \frac{2t^2}{m^2}), \\ V_5^{(2)} &= -(u \wedge t)^2 \mathcal{D} b / m^2 \doteq 0, \\ V_6^{(2)} &= \mathcal{D} (-B + b \frac{t^2}{m^2}) \doteq \frac{8}{m^2} \mathcal{D} t^2, \end{aligned} \right\} \quad (5.9)$$

$$\mathcal{D} = \frac{\bar{u}^{(1)} \cdot \bar{u}^{(2)}}{4t^2 + \mu^2} \frac{m^2}{4k^0 k'^0 (k'^0 + m)(k^0 + m)}$$

$$\doteq \bar{u}^{(1)} \cdot \bar{u}^{(2)} / \{16 m^2 (4t^2 + \mu^2)\},$$

$$B = (k'^0 - k^0)^2 / m^2 \doteq 0,$$

$$A = \frac{1}{m^2} (k'^0 + k^0 + 2m)^2 \doteq 16 + \frac{4(\frac{t^2}{m^2} + k'^0 k^0)}{m^2},$$

$$b = \frac{4(k'^0 + k^0 + 2m)}{k'^0 + k^0} \doteq 8.$$

We have taken the convention that  $\underline{u}^{(1)}$  and  $\underline{u}^{(2)}$  are to be taken between the nucleons  $k$  and  $k'$ . Hoshizaki and Machida<sup>(24)</sup> made too drastic an approximation to  $\underline{u}$  and as a consequence their expression for  $V_A^{(2)}$  contained  $8\frac{k^2}{m^2}$  instead of  $-2\frac{k^2}{m^2}$ .

(2) Contribution from the "direct" graphs,  $G^A$  :-

The fourth order contributions from (5.4) consist of graphs representing renormalization to the second order one-meson-exchange graphs, and the "direct" and "crossed" graphs which are

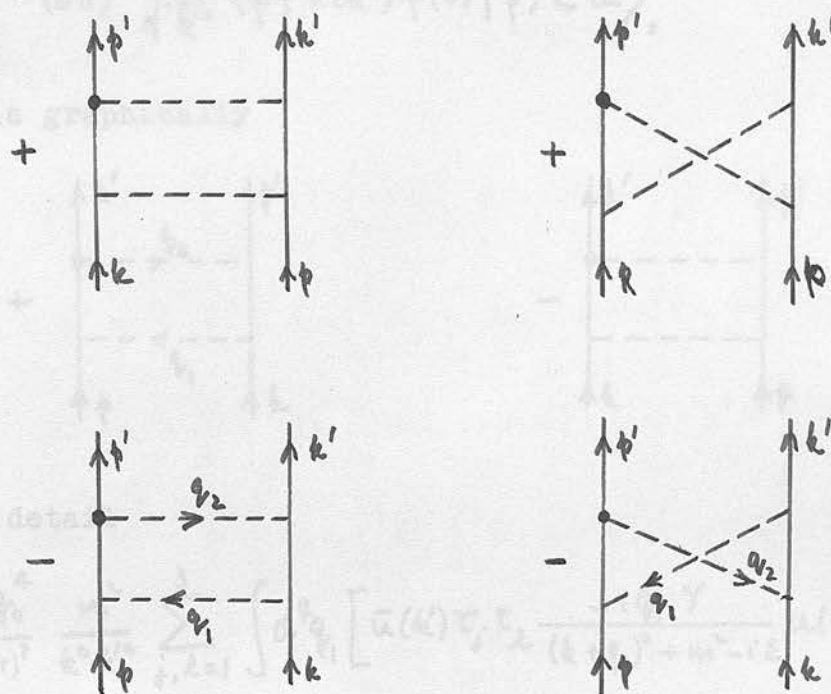


Fig. IV

The renormalization graphs we took account of in the last

The average  $G^A$  of (5.10) and (5.12) is symmetric with



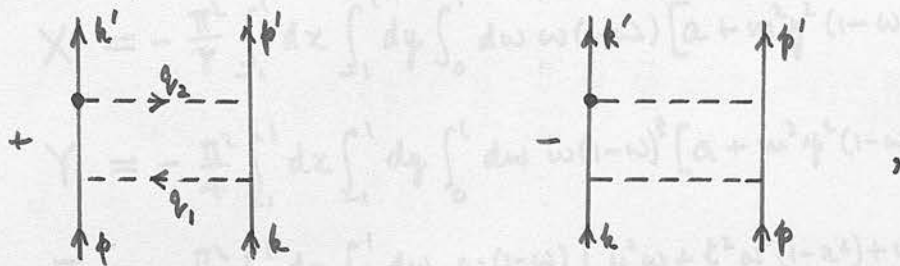
section. Here we deal with the "direct" graphs, the pair on the left of Fig. IV. Their combined contribution is

$$\frac{i q_0^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} \sum_{j,l=1}^3 \int d^4 q_l \left[ \bar{u}(p') \frac{-i \gamma \cdot q_l}{(p+q_l)^2 + m^2 - i\varepsilon} \tau_j \tau_l u(p) \frac{1}{q_l^2 + \mu^2 - i\varepsilon} \times \right. \\ \left. \times \bar{u}(k') \frac{i q_l \cdot \gamma}{(k-q_l)^2 + m^2 - i\varepsilon} \tau_j \tau_l u(k) \frac{1}{(k'+q_l-k)^2 + \mu^2 - i\varepsilon} - (p \leftrightarrow k) \right]. \quad (5.10)$$

This is not symmetric with respect to  $\sigma^{(n)}$  and  $\sigma^{(a)}$  as shown in (A1.1), (A1.3) and (A1.6) in Appendix 1. However neither is the corresponding expression from

$$(2\pi)^{3/2} \sqrt{\frac{m}{k^0}} \langle p' | \bar{u}(k') f(0) | p, k \text{ in} \rangle, \quad (5.11)$$

which is graphically



and in detail

$$\frac{i q_0^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} \sum_{j,l=1}^3 \int d^4 q_l \left[ \bar{u}(k') \tau_j \tau_l \frac{-i q_l \cdot \gamma}{(k+q_l)^2 + m^2 - i\varepsilon} u(k) \frac{1}{q_l^2 + \mu^2 - i\varepsilon} \times \right. \\ \left. \times \bar{u}(p') \tau_j \tau_l \frac{i q_l \cdot \gamma}{(p-q_l)^2 + m^2 - i\varepsilon} u(p) \frac{1}{(p'+q_l-p)^2 + \mu^2 - i\varepsilon} - (p \leftrightarrow k) \right]. \quad (5.12)$$

The average  $G^A$  of (5.10) and (5.12) is symmetric with

respect to  $\mathcal{I}^{(1)}$  and  $\mathcal{I}^{(2)}$ , and is actually the contribution from these "direct" graphs to the potential. We work this out in detail in Appendix 1 and only write down the result here:

$$G^A = \frac{q_0^4}{(2\pi)^7} (3 - 2\mathcal{I}^{(1)} \cdot \mathcal{I}^{(2)}) \frac{1}{m^2} \left[ \sum_{i=1}^6 G_i^A \mathcal{U}_i - (\mathcal{I} \leftrightarrow \mathcal{U}) p_\tau p_\sigma \right], \quad (5.13)$$

$$G_1^A \doteq m^2 F + \mathcal{I}^2 (X + Y \mathcal{U}^2) - \eta^2 \mathcal{U} \cdot \mathcal{I} G + (\mathcal{U} \cdot \mathcal{I})^2 (H + G),$$

$$G_2^A \doteq \frac{1}{2} F + X + \mathcal{U}^2 Y - \mathcal{U} \cdot \mathcal{I} G \left(1 + \frac{\mathcal{U}^2}{2m^2}\right) + (\mathcal{U} \cdot \mathcal{I})^2 (H + G) \frac{1}{2m^2},$$

$$G_3^A \doteq -\frac{1}{4m^2} F + \frac{1}{2m^2} X - Y,$$

$$G_4^A \doteq X,$$

$$G_5^A \doteq -\mathcal{I}^2 X,$$

$$G_6^A \doteq 0;$$

with

$$\left. \begin{aligned} X &= -\frac{\pi^2}{8} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \, w(1-w) [a + m^2 y^2 (1-w)^2 - i\varepsilon]^{-1}, \\ Y &= -\frac{\pi^2}{4} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \, w(1-w)^3 [a + m^2 y^2 (1-w)^2 - i\varepsilon]^{-2}, \\ F &= -\frac{\pi^2}{4} \int_{-1}^1 dx \int_0^1 dw \, w(1-w) [\mu^2 w + \mathcal{I}^2 w^2 (1-x^2) + m^2 (1-w)^2]^{-1}, \\ G &= -\frac{\pi^2}{4} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \, w(1-w)^2 (1+xw) [a + m^2 y^2 (1-w)^2 - i\varepsilon]^{-2}, \\ H &= -\frac{\pi^2}{4} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \, w^3 (1-w)^2 (1+x) [a + m^2 y^2 (1-w)^2 - i\varepsilon]^{-2}, \\ a &= \mu^2 w - \eta^2 (1-w)^2 + (1-x^2) w^2 \mathcal{I}^2 + 2\mathcal{U} \cdot \mathcal{I} (1-w)(1+xw), \\ \eta^2 &= \frac{1}{2} (\mathcal{I}^2 + \mathcal{U}^2). \end{aligned} \right\} \quad (5.14)$$

and

\* In this and the following two sections our expressions are so similar to those of Charap and Tausner that we decide to follow their terminology and manipulations very closely.

(3) Contribution from the "crossed" graphs,  $V^B$  :-

The "crossed" graphs from (5.4) are the pair on the right of Fig. IV. Their combined contribution to the potential is

$$\begin{aligned} & \frac{i q_0^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} \sum_{j,l=1}^3 \int d^4 q_1 \left[ \bar{u}(p') \frac{i \gamma \cdot q_1}{(p-q_1)^2 + m^2 - i\epsilon} \tau_j \tau_l u(p) \frac{1}{q_1^2 + \mu^2 - i\epsilon} \times \right. \\ & \left. \times \bar{u}(k') \frac{i \gamma \cdot q_1}{(k'-q_1)^2 + m^2 - i\epsilon} \tau_l \tau_j u(k) \frac{1}{(k'-q_1-k)^2 + \mu^2 - i\epsilon} - (p \leftrightarrow k) \right]. \end{aligned} \quad (5.15)$$

As is shown in Appendix 2, this expression is approximately the same as the corresponding expression from (5.11). The contribution to the potential from all the "crossed" graphs is thus

$$V^B = \frac{q_0^4}{(2\pi)^7} \frac{1}{m^2} (3 + 2 \epsilon^{(1)} \cdot \epsilon^{(2)}) \left[ \sum_{i=1}^6 V_i^B \mathcal{O}_i - (t \leftrightarrow u) P_t P_u \right] \quad (5.16)$$

If we put this expression into (5.2), the lowest order expression of the latter is obtained when the first matrix element

$$V_1^B \doteq m^2 F - m^4 S,$$

element of the latter is obtained when the first matrix element

$$V_2^B \doteq \frac{3}{2} F - \frac{m^2}{2} S,$$

graph:

$$V_3^B \doteq \frac{1}{4m^2} F - \frac{1}{12} S,$$

$$V_4^B \doteq F,$$

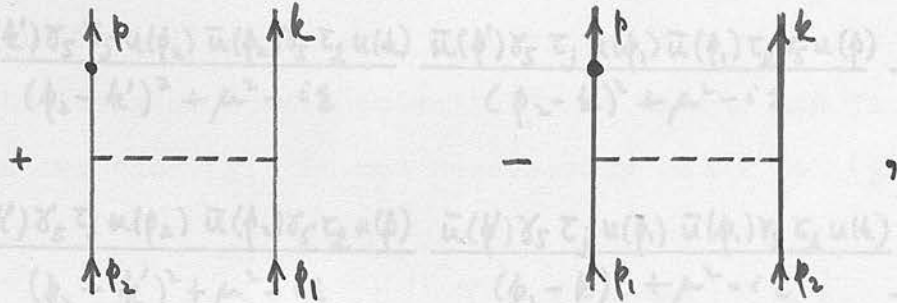
$$V_5^B \doteq -t^2 F,$$

$$V_6^B \doteq 0;$$

with 
$$S = -\frac{\pi^2}{2} \int_0^1 dx \int_0^1 d\omega \omega (1-\omega)^3 [\mu^2 \omega + k^2 \omega^2 (1-x^2) + m^2 (1-\omega)^2]^{-2} \quad (5.17)$$

(3) The iterated one-meson-exchange graphs, I :-

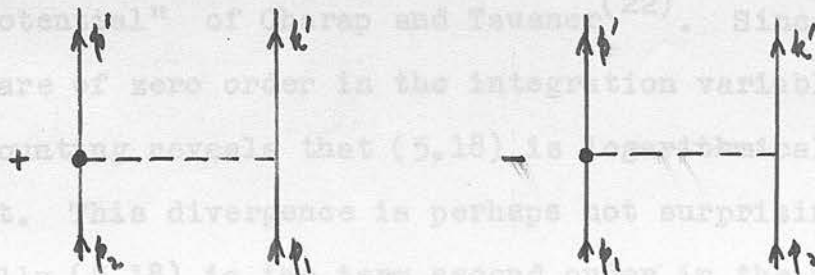
The lowest order contribution of the second class to (5.3) is graphically



and in detail

$$\frac{g_0^2}{(2\pi)^{9/2}} \frac{m^2}{p_1^0 k^0} \sum_{j=1}^3 \left[ \frac{\bar{u}(k) \gamma_5 \tau_j u(p_2)}{[(p_2 - k)^2 + \mu^2 - i\epsilon]} \frac{\bar{u}(p) \gamma_5 \tau_j u(p_1)}{2p_1^0 - 2k^0 + i\epsilon} - (p_1 \leftrightarrow p_2) \right].$$

If we put this expression into (5.2), the lowest order expression of the latter is obtained when the first matrix element in (5.2) is also represented by its lowest order graph:





which gives

$$\frac{g_0^2}{(2\pi)^{9/2}} \frac{m^2}{k^0 p_1^0} \sum_{j=1}^3 \left[ - \frac{\bar{u}(k') \gamma_5 \tau_j u(p_2) \bar{u}(p') \gamma_5 \tau_j u(p_1)}{(p_2 - k')^2 + \mu^2 - i\varepsilon} + (p_1 \leftrightarrow p_2) \right],$$

Thus putting these two expressions into (5.2) gives

$$\begin{aligned} & - \frac{g_0^4}{2(2\pi)^6} \frac{m^4}{k^0 k'^0 p_1^0 p_2^0} \int d^3 p_1 d^3 p_2 \delta(p_1 + p_2) \frac{2}{2p_1^0 - 2k^0 - i\varepsilon} \sum_{j,l=1}^3 \times \\ & \times \left[ \frac{\bar{u}(k') \gamma_5 \tau_j u(p_2) \bar{u}(p_2) \gamma_5 \tau_l u(k)}{(p_2 - k')^2 + \mu^2 - i\varepsilon} \frac{\bar{u}(p') \gamma_5 \tau_j u(p_1) \bar{u}(p_1) \tau_l \gamma_5 u(p)}{(p_2 - k)^2 + \mu^2 - i\varepsilon} - \right. \\ & \left. - \frac{\bar{u}(k') \gamma_5 \tau_j u(p_2) \bar{u}(p_2) \gamma_5 \tau_l u(p)}{(p_2 - k')^2 + \mu^2 - i\varepsilon} \frac{\bar{u}(p') \gamma_5 \tau_j u(p_1) \bar{u}(p_1) \gamma_5 \tau_l u(k)}{(p_1 - k)^2 + \mu^2 - i\varepsilon} \right]. \end{aligned} \quad (5.18)$$

Bearing in mind that we are working in the C-M system, the second term of (5.18) can be obtained from the first by the interchange  $p \leftrightarrow k$ . The corresponding contributions to

$V(k'; k)$  from  $\langle k', p'; k, p \rangle$ , etc., can be obtained from (5.18) by the interchanges  $p \leftrightarrow k$ , or  $p' \leftrightarrow k'$  or both. Their combined contribution I to  $V(k'; k)$  is exactly equal to (5.18), which corresponds to the "iterated Yukawa potential" of Charap and Tausner<sup>(22)</sup>. Since the spinors are of zero order in the integration variables, a simple counting reveals that (5.18) is logarithmically divergent. This divergence is perhaps not surprising, for essentially (5.18) is the term second order in the T-matrix

of our Low equation, with the T-matrix elements approximated to the lowest order in  $g$ . This last mentioned approximation is not at all good because at least one of the states involved in each T-matrix element is carried to arbitrarily high energy, and hence the divergence. This does not matter, as was pointed out in the last chapter. We will approximate the various expressions in (5.18) as if all the three-vectors had magnitudes less than or equal to  $\mu$ , and then take the resulting expression to be valid for all energies. Thus we get the linearly divergent "iterated Yukawa potential" of Charap and Tausner, except in our case  $|\underline{k}|$  is not necessarily equal to  $|\underline{k}'|$ :

$$I = \frac{g_0^4}{(2\pi)^7} \frac{1}{m^2} (3 - 2\underline{\epsilon}^{(1)} \cdot \underline{\epsilon}^{(2)}) [J - (\underline{\epsilon} \leftrightarrow \underline{u}) P_\epsilon P_\sigma], \quad (5.19)$$

with

$$J = \frac{\pi}{8m} \int_{-\infty}^{\infty} \frac{d^3 p_2}{p_2^2 - \underline{k}^2 - i\epsilon} \left[ \frac{(\underline{k}' - \underline{p}_2) \cdot (\underline{p}_2 - \underline{k}) + i\sigma^{(1)} \cdot (\underline{k}' - \underline{p}_2) \wedge (\underline{p}_2 - \underline{k})}{(\underline{p}_2 - \underline{k}')^2 + \mu^2} \times \right. \\ \left. \times \frac{(\underline{p}' + \underline{p}_2) \cdot (\underline{p}_2 + \underline{p}) + i\sigma^{(2)} \cdot (\underline{p}' + \underline{p}_2) \wedge (\underline{p}_2 + \underline{p})}{(\underline{p}_2 - \underline{k})^2 + \mu^2} \right]. \quad (5.19a)$$

In Appendix 3 we show that we have

$$J = \sum_{i=1}^6 J_i Q_i, \quad (5.20)$$

From now on the letter  $t$  denotes  $|\underline{t}|$ .

with

$$J_1 = W - \underline{t}^2 (\underline{u}^2 Y_s + X_s) + \eta^2 \underline{u} \cdot \underline{t} G_s - (\underline{u} \cdot \underline{t})^2 (G_s + H_s + L_s),$$

$$J_2 = -X_s - \underline{u}^2 Y_s + G_s \underline{u} \cdot \underline{t},$$

$$J_3 = Y_s,$$

$$J_4 = -X_s,$$

$$J_5 = \underline{t}^2 X_s,$$

$$J_6 = 0,$$

where

$$W = (2\pi)^2 \frac{Q}{8m} - \frac{(2\pi)^3}{64m} \left[ 2\mu + \left( \frac{\mu^2}{t} + \frac{2\underline{u} \cdot \underline{t}}{t} \right) \sin^{-1} \frac{t}{\sqrt{t^2 + \mu^2}} \right],$$

Q being an arbitrarily large number,

$$\left. \begin{aligned} X_s &= -\frac{\pi^2}{8m} \int_{-1}^1 dx \int_0^1 \omega d\omega \int_{-\infty}^{\infty} dq [q^2 + a - i\varepsilon]^{-1}, \\ Y_s &= -\frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 \omega (1-\omega)^2 d\omega \int_{-\infty}^{\infty} dq [q^2 + a - i\varepsilon]^{-2}, \\ G_s &= -\frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 \omega (1-\omega)(1+x\omega) d\omega \int_{-\infty}^{\infty} dq [q^2 + a - i\varepsilon]^{-2}, \\ H_s &= -\frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 \omega^2 (1-\omega)(1+x) d\omega \int_{-\infty}^{\infty} dq [q^2 + a - i\varepsilon]^{-2}, \\ L_s &= -\frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 \omega^3 (1+x)^2 d\omega \int_{-\infty}^{\infty} dq [q^2 + a - i\varepsilon]^{-2} \end{aligned} \right\} \quad (5.20a)$$

From now on the letter  $t$  denotes  $|\underline{t}|$ . Since we are most

(4) Contribution from the deuteron states,  $V^D$  :-

There is still no generally accepted way of handling the bound states in Q.F.T. Fortunately a lot of information has been gathered about the deuteron, and this can be handled more easily in the formalism of non-relativistic Quantum Mechanics. Since we are only interested in the low energy region, use of a non-relativistic formalism will suffice.

The expression under consideration is

$$\begin{aligned} V^D &= \sum_B T(\underline{k}', -\underline{k}'; B) \chi(\underline{k}, -\underline{k}; B) \\ &= \sum_B (E_B - 2E_{\underline{k}}) \chi(\underline{k}', -\underline{k}'; B) \chi^*(\underline{k}, -\underline{k}; B), \end{aligned} \quad (5.22)$$

and in Quantum Mechanics this becomes in matrix form,

$$\begin{aligned} V^D &= (-B - \frac{k^2}{m}) \frac{1}{(2\pi)^3} \frac{1}{2} (1 + P_\sigma) \int d^3x e^{-i\underline{k} \cdot \underline{x}} \psi_B(\underline{x}) \frac{1}{2} (1 + P_\sigma) \times \\ &\times \int d^3x' e^{+i\underline{k} \cdot \underline{x}'} \psi_B^\dagger(\underline{x}') \frac{1}{2} (1 + P_\sigma) \frac{1}{2} (1 + P_\tau), \end{aligned} \quad (5.21)$$

where  $B$  is the binding energy of the deuteron, and

$$\psi_B(\underline{x}) = \frac{v(r)}{r} + \frac{w(r)}{r} S_{12}(\hat{r}),$$

with  $S_{12}(\hat{r}) = 3 \hat{\sigma}^{(1)} \cdot \hat{r} \hat{\sigma}^{(2)} \cdot \hat{r} - \hat{\sigma}^{(1)} \cdot \hat{\sigma}^{(2)}$ . Current estimates<sup>(25)</sup> of the D-state admixture, that is, the term containing  $w(r)$  based on the anomalous magnetic moment of the deuteron, range from 3% to 7% of the total  $\psi_B(\underline{x})$ . Since we are most



interested in the low energy region where  $|\underline{k}| < \mu$ ,  $|\underline{k}'| < \mu$ , we will ignore  $\omega(\tau)$ . We have then essentially a scalar deuteron, in which event Blankenbecler and Cook<sup>(20)</sup> showed that within the limit of certain approximations, we can derive the Hulthen wavefunction for the deuteron from Q.F.T. and

$$\int d^3x \psi_b(z) e^{-i\underline{k}' \cdot \underline{z}} \doteq 48NmB \frac{4\pi}{(\underline{k}'^2 + mB)(\underline{k}'^2 + 49mB)}, \quad (5.22)$$

with

$$4\pi N^2 \int_0^\infty d\tau (e^{-\sqrt{mB}\tau} - e^{-7\sqrt{mB}\tau})^2 = 1,$$

or

$$N^2 \doteq \frac{3\sqrt{mB}}{4\pi}. \quad (5.23)$$

Substitution of (5.22) into (5.21) then gives

$$V^D = -\frac{1}{2}(1+P_\sigma)\frac{1}{2}(1+P_\tau)D(\underline{k}', \underline{k}'), \quad (5.24)$$

where

$$D(\underline{k}', \underline{k}') = \frac{N^2}{(2\pi)^3 m} \frac{4^2 (48mB)^2}{(\underline{k}'^2 + 49mB)(\underline{k}'^2 + 49mB)(\underline{k}'^2 + mB)},$$

or

$$V^D = -\frac{1}{2}(1-P_\tau) \left[ \left\{ \frac{3}{8} D(\underline{k}', \underline{k}') Q_1 + \frac{1}{8} D(\underline{k}', \underline{k}') Q_5 \right\} - (\underline{k} \leftrightarrow \underline{k}') P_\tau P_\sigma \right]. \quad (5.25)$$

Thus  $V^D$  is highly energy dependent because  $D(\underline{k}', \underline{k}')$  is.

If we use the dominant term in  $V^{(2)}$  as standard, then

$D(\underline{k}', \underline{k}')$  varies from about unity at  $\underline{k}' = \underline{k}' = 0$  to  $1/10$  at  $\underline{k}' = \underline{k}' = \mu^2$ . Hence if the potential is to be

energy independent in the sense that the coefficients of  $\mathcal{O}_i$  are, the energy dependence of  $D(k^1, k^2)$  can be cancelled only by  $G^A$  and  $I$ , if at all. We will proceed then to see the combined effect of  $G^A$  and  $I$ .

(5) The combined contribution of  $G^A$  and  $I$ ,  $V^A$  :-

In (5.14), if we make the substitution

$$q = m y (1 - \omega),$$

then  $X$  and  $Y$  become

$$X = -\frac{\pi^2}{8m} \int_1^1 dx \int_0^1 d\omega \omega \int_{-m(1-\omega)}^{m(1-\omega)} dq [q^2 + a - i\varepsilon]^{-1},$$

$$Y = -\frac{\pi^2}{4m} \int_1^1 dx \int_0^1 d\omega \omega \int_{-m(1-\omega)}^{m(1-\omega)} dq [q^2 + a - i\varepsilon]^{-2} (1-\omega)^2.$$

Thus

$$V^A = G^A + I$$

$$= \frac{g_0^4}{(2\pi)^7} \frac{1}{m^2} (3 - 2\varepsilon^{(1)} \cdot \varepsilon^{(2)}) \left[ \sum_{i=1}^6 V_i^A \mathcal{O}_i - (\underline{t} \leftrightarrow \underline{u}) P_c P_\sigma \right];$$

$$V_1^A = W + m^2 F + t^2 (u^2 Y_c + X_c) - \eta^2 \underline{u} \cdot \underline{t} G_c + (\underline{u} \cdot \underline{t})^2 (G_c + H_c + L_c), \quad (5.25)$$

$$V_2^A = \frac{1}{2} F + (X_c + u^2 Y_c) - \underline{u} \cdot \underline{t} \left[ G_c + \frac{u^2}{2m^2} (G_c + G_s) \right] + \frac{(\underline{u} \cdot \underline{t})^2}{2m^2} (H_c + G_c + H_s + G_s),$$

$$V_3^A = -\frac{1}{4m^2} F + \frac{1}{2m^2} (X_c + X_s) - Y_c,$$

$$V_4^A = X_c,$$

$$V_5^A = -t^2 X_c,$$

$$V_6^A = 0;$$

(5.26)

with

$$X_c = \frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 dw \omega \int_{m(1-w)}^{\infty} dq (q^2 + a)^{-1}$$

$$\doteq \frac{\pi^2}{4m} \int_{-1}^1 dz \int_0^1 dw \omega \int_{m(1-w)}^{\infty} dq [\mu^2 \omega + t^2 \omega^2 (1-z^2) + q^2]^{-1},$$

$$Y_c = \frac{\pi^2}{2m} \int_{-1}^1 dx \int_0^1 dw \omega (1-w)^2 \int_{m(1-w)}^{\infty} dq [q^2 + a]^{-2}$$

$$\doteq \frac{\pi^2}{2m} \int_{-1}^1 dz \int_0^1 dw \omega (1-w)^2 \int_{m(1-w)}^{\infty} dq [q^2 + \mu^2 \omega + t^2 \omega^2 (1-z^2)]^{-2},$$

$$G_c = \frac{\pi^2}{2m} \int_{-1}^1 dz \int_0^1 dw \omega (1-w)(1+zw) \int_{m(1-w)}^{\infty} dq (q^2 + a)^{-2}$$

$$\doteq \frac{\pi^2}{2m} \int_{-1}^1 dz \int_0^1 dw \omega (1-w) \int_{m(1-w)}^{\infty} dq [q^2 + \mu^2 \omega + t^2 \omega^2 (1-z^2)]^{-2},$$

and

$$H_c = \frac{\pi^2}{2m} \int_{-1}^1 dz \int_0^1 dw \omega^2 (1-w)(1+z) \int_{m(1-w)}^{\infty} dq (q^2 + a)^{-2}$$

$$= \frac{\pi^2}{2m} \int_{-1}^1 dx \int_0^1 dw \omega^2 (1-w) \int_{m(1-w)}^{\infty} dq [q^2 + \mu^2 \omega + t^2 \omega^2 (1-z^2)]^{-2}.$$

In Appendix 4, we show that we can ignore all the terms involving  $H_s$ ,  $G_s$ ,  $H_c$ ,  $G_c$  and  $Y_c$ , and that we can ignore  $V_3^A$  compared with the other terms in (5.25). Thus we have

$$\left. \begin{aligned} V_1^A &\doteq W + m^2 F + t^2 X_c, \\ V_2^A &\doteq \frac{1}{2} F + X_c, \\ V_3^A &\doteq 0, \\ V_4^A &= X_c, \\ V_5^A &= -t^2 X_c, \\ V_6^A &\doteq 0. \end{aligned} \right\} \quad (5.26)$$

(5.26) differs from the set obtained by Charap and Tausner only in that  $W$  in our case contain a ~~time~~-dependent term.

# (6) Spectral functions:-

We can now proceed to express our potential as the superposition of Yukawa potentials by first putting  $F$ ,  $X_c$ ,  $Y_c$ ,  $U$  into the form

$$F = \frac{1}{(2\pi)^3} \int_{4\mu^2}^{\infty} d\sigma^2 \frac{f(\sigma^2)}{\sigma^2 + 4t^2}, \quad (5.27)$$

etc. With the exception of  $V_1^A$  due to  $W$  and  $V_3^B$ ,  $V_3^A$ ; the following expressions are identical to those of Charap and Tausner. (22) We write them down for completeness. If we put

$$\tau = \frac{\sigma}{\mu},$$

$$q^2 = \tau^2 - 4,$$

$$\omega^2 = 4M^2 - \tau^2,$$

$$M = \frac{m}{\mu},$$

where  $W(0, t)$  is of course infinite. We will write  $V_1^A$  in a form, and for comparison do the same thing for  $V_1^B$ . Then we have

$$f(\sigma^2) = - \frac{(2\pi)^5}{4\mu^2 \tau \omega^2} \left( q - \frac{q^2 + 2}{|\omega|} \Phi \right), \quad (5.28)$$

$$x(\sigma^2) = \frac{(2\pi)^5}{4\mu^2 M \tau^3} \left\{ (q^2 + 2) \left( \frac{\pi}{4} - \frac{M}{|\omega|} \Phi \right) - \Theta \right\}, \quad (5.29)$$



$$\Delta(\sigma^2) = \frac{(2\pi)^5}{\mu^4 \tau \omega^4} \left[ \frac{12(q^2+2)}{|\omega|} \Phi - q \left\{ 8 + \frac{(q^2+2)^2}{1+M^2 q^2} \right\} \right], \quad (5.30)$$

$$\begin{aligned} \gamma(\sigma^2) = \frac{(2\pi)^5}{\mu^4 M \tau^3} \left\{ \frac{\tau^2 \Theta}{4} + \frac{M q}{\omega^2} + \frac{q^2+2}{\tau^2} \frac{2\omega^2 - \tau^2}{\omega^2} \frac{M}{|\omega|} \Phi - \right. \\ \left. - \frac{q^2+2}{\tau^2} \left( \frac{\pi}{2} + \Theta \right) \right\}, \end{aligned} \quad (5.31)$$

where

$$\Phi = \cos^{-1} \left\{ \frac{q^2+2}{2\sqrt{1+M^2 q^2}} \right\} \quad \text{if} \quad \omega^2 \geq 0,$$

$$= \log \left\{ \frac{q^2+2 + q\sqrt{-\omega^2}}{2\sqrt{1+M^2 q^2}} \right\} \quad \text{if} \quad \omega^2 \leq 0,$$

and

$$\Theta = \cot^{-1}(Mq).$$

We put  $W$  is a subtracted form:

$$\begin{aligned} W(\underline{t}, \underline{u}) = W(0, \underline{u}) - \frac{(2\pi)^3}{64m} \left[ 2 \underline{u} \cdot \underline{t} \int_{4\mu^2}^{\infty} \frac{d\sigma^2}{\sigma(\sigma^2+4t^2)} - \right. \\ \left. - 4\mu^2 t^2 \int_{4\mu^2}^{\infty} \frac{d\sigma^2}{\sigma^3(\sigma^2+4t^2)} \right], \end{aligned} \quad (5.33)$$

where  $W(0, \underline{u})$  is of course infinite. We will write  $V_1^A$  in a subtracted form, and for comparison do the same thing for  $V_1^B$ .

Then

$$V_1^{A,B} = \frac{(2\pi)^2}{\mu^2} \int_{-\infty}^{\infty} \frac{d\sigma^2}{4\mu^2 \sigma^2 + 4t^2} \zeta_1^{A,B}(\sigma^2, u, t) + \text{constant}^{A,B},$$

$$V_{2,4}^{A,B} = \frac{(2\pi)^2}{\mu^2} \int_{-\infty}^{\infty} \frac{d\sigma^2}{4\mu^2 \sigma^2 + 4t^2} \zeta_{2,4}^{A,B}(\sigma^2),$$

$$V_3^{A,B} = \frac{(2\pi)^2}{\mu^4} \int_{-\infty}^{\infty} \frac{d\sigma^2}{4\mu^2 \sigma^2 + 4t^2} \zeta_3^{A,B}(\sigma^2),$$

$$V_5^{A,B} = \frac{(2\pi)^2}{\mu^2} t^2 \int_{-\infty}^{\infty} \frac{d\sigma^2}{4\mu^2 \sigma^2 + 4t^2} \zeta_5^{A,B}(\sigma^2),$$

(5.32)

with

$$\zeta_1^A = \frac{\mu^2}{(2\pi)^5} \left\{ \chi t^2 - \frac{4t^2 M^2}{t^2} f - \frac{(2\pi)^6}{32} \left( \frac{u \cdot t}{\mu t} - \frac{2t^2}{\mu t^3} \right) \right\},$$

$$\zeta_1^B = -\frac{t^2 \mu^2}{(2\pi)^5} \frac{4M^2}{t^2} (M^2 \mu^2 \rho - f),$$

$$\zeta_2^A = \frac{\mu^2}{(2\pi)^5} \left( \chi + \frac{1}{2} f \right),$$

$$\zeta_2^B = \frac{\mu^2}{(2\pi)^5} \frac{1}{2} (M^2 \mu^2 \rho - 3f),$$

$$\zeta_3^A \doteq 0,$$

$$\zeta_3^B = -\frac{\mu^2}{(2\pi)^5} \frac{1}{12} \left( -\mu^2 \rho + \frac{3}{4} \frac{f}{M^2} \right),$$

$$\zeta_4^A = -\zeta_5^A = \frac{\mu^2}{(2\pi)^5} \chi,$$

$$\zeta_4^B = -\zeta_5^B = -\frac{\mu^2}{(2\pi)^5} f.$$

(5.33)

(7) The potential  $\mathcal{U}$  :-

The potential  $\mathcal{U}$  defined in (4.11b) can be handled in exactly the same manner. To fourth order in  $g$ , it turns out that we can obtain  $\mathcal{U}$  by the replacement

$$\underline{u.t} \longleftrightarrow - \underline{u.t}$$

in  $V$ . This means  $\mathcal{U}$  and  $V$  are not equal in perturbation theory, at least to fourth order.

## CHAPTER 6

### DISCUSSION.

We devised in Chapter 4 a method by means of which a transition from Q.F.T. to a potential theory for a system of two particles can be effected without assuming much beyond the general postulates of Axiomatic Q.F.T. This is the decisive advantage of our method over the ones outlined in Chapter 1.

There, the first major step was the establishment of the "first" Lippmann-Schwinger equation (4.10a) which is exact if the energy of the scattering state is below the pion production threshold. Jauch and Rohrlich<sup>(27)</sup>, and Heisenberg<sup>(28)</sup> claimed to have proved the same equation in Q.F.T. for all energies, but throughout their proofs, they assumed the possibility of using the eigenstates of the free Hamiltonian as a basis of representation. Moreover, their formalism then led to the "second" Lippmann-Schwinger equation (2.9a) in which the matrix elements of the potential were replaced by those of the interaction Hamiltonian between two free states. Since for trilinear interactions the latter matrix elements vanish due to the violation of conservation of energy-momentum, their equivalent of (2.9a) means that the reaction matrix vanishes identically!

From (4.10a) we obtained the Low equation (4.11a), exact for the same energy range. This latter should be compared with the one obtained by Low<sup>(29)</sup> whose method only works in the



peculiar case of pion-nucleon scattering. His equation is approximate even in the low energy region. Examination of (4.11a) showed that we have much freedom to assign values to the scattering-matrix elements involving high energy scattering states. In particular we can demand the D-matrix elements between two-particle states to have the same functional form as those of them which have both energies below the threshold, which course of action we adopted in Chapter 5, and in Chapter 3 albeit only on the energy-shell. Under this circumstance the T-matrix elements on the energy-shell defined in both Chapters for a potential theory should be equal in the case of boson-boson scattering, and hence if elements of the potential on the energy-shell obtained in these Chapters differ, it must be due to the fact that the T-matrix elements are split into two parts, the potential and the rest, differently.

Now in the case of non-identical boson-boson scattering, a moment's reflection will confirm that the potential  $V$  is then

$$V = V^{(1)} + V^A + V^B + V^D, \quad (6.1)$$

with  $V^{(1)}$ ,  $V^A$ ,  $V^B$  the same as in Reference 10, i.e., as would be obtained from (3.22), and

$$V^D = -D(k^1, k^2)$$

where  $D(k^1, k^2)$  is as defined in (5.24). Thus (6.1) differs from the potential defined in Chapter 3 only in the

estimate of the contribution from the "deuteron". This close similarity within the framework of perturbation theory and approximation to fourth order in the coupling constant, leads us to suspect that in these two methods of defining the potential, the split of the T-matrix elements on the energy-shell is probably the same, and that their difference arises from the shortcoming of the method of calculation.\* This was the reason why in Chapter 5 we did not press on to calculate numerically and compare with experimental results.

At present, our potential theory suffers from two defects. The first one is that the wavefunctions of the theory may not be orthonormal. The second is that we have no reliable method of calculating the potential defined in it. To a great extent these defects are due to the fact that in Q.F.T., ~~still wanting~~ ~~is~~ a satisfactory way of estimating, never mind calculating, most expressions arising ~~in it~~ is still wanting.

In the last few years efforts have been concentrated in investigating T-matrix elements on the energy-shell, using analyticity and unitarity as the main tools. In a few cases dispersion relations in the energy variable come out of the complicated wash. Some people took as fundamental the double

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\* It has been generally assumed that higher order graphs will give rise to forces of shorter ranges. Matthews<sup>(32)</sup> recently showed, however, that the force due to three-pion exchange seems to have much longer range than that due to two-pion exchange.

representation for the T-matrix elements conjectured by Mandelstam. The latest work in this line of development was the paper by Chew and Frautschi<sup>(30)</sup> who developed the so-called "strip approximation" which seems good for all physical energy provided the magnitude of momentum transfer is less than  $4\mu$ . For This range of value is precisely what we are interested in for calculating our potential. Unfortunately the feature that makes this approximation possible has no analogue at all for matrix elements off the energy-shell. As matter stands to-day, the most hopeful direction in which to look for a solution of our problem still seems to lie in dispersion relations, although the problem of establishing these relations will be as likely as not more difficult because three, instead of the usual two, independent variables are involved, and the matrix elements are no longer relativistically invariant because of the non-conservation of energy.

Very recently, Fivel<sup>(31)</sup> took a preliminary step in this direction. Working in the centre-of-mass system and considering the case of boson-boson scattering in a potential theory, he adopted as the definition of the T-matrix element  $T^{(-)}$  in (2.16) and proved, under the assumption of space- and time-reversal invariance, that  $T^{(-)}(\underline{k}'; \underline{k})$  obeys a Mandelstam representation in  $\underline{k}^2$  and  $(\underline{k}' - \underline{k})^2$  if  $\underline{k}^2$  is held fixed, and that the generalization of the unitarity condition is

$$\text{Im } T^{(-)}(\underline{k}', \underline{k}) = \sum_{\underline{k}_n^2 = \underline{k}^2} T^{(-)}(\underline{k}'; \underline{k}_n) T^{(-)*}(\underline{k}; \underline{k}_n). \quad (6.2)$$

where



(6.2) together with a one-dimensional dispersion relation for  $T^{(-)}$  provides a basis for an iterative procedure so that we get an expression giving the off-shell elements in terms of the ones on the energy-shell.

The analogue of (6.2) can be demonstrated in Q.F.T. For simplicity we consider boson-boson scattering. Let us consider

$$\xi = -i (2\pi)^{3/2} \sqrt{\frac{1}{2p^0}} \langle k' | j(0) | p, k \text{ in} \rangle,$$

where  $j(x) = (\partial^\mu \partial_\mu + m^2)\phi(x)$ ,  $\phi(x)$  being the field operator. The usual manipulations give

$$\begin{aligned} \xi = \frac{-i}{2\sqrt{p^0 p'^0}} \int d^4x e^{i(p + \frac{1}{2}k - \frac{1}{2}k') \cdot x} \langle k' | \{ \theta(-x) [j(-\frac{x}{2}), j^+(\frac{x}{2})] - \\ - \delta(x^0) [j(-\frac{x}{2}), \dot{\phi}(\frac{x}{2})] \} | k \rangle. \end{aligned} \quad (6.3)$$

If we assume that the interaction is of the trilinear type, microscopic causality tells us that the matrix element of the equal time commutator depends only on  $(k' - k)^2$  and is real. It will not complicate our further argument and will therefore be neglected in the following.

Using

$$\theta(x) = \frac{1}{2} [1 + \varepsilon(x)], \quad \varepsilon(x) = \begin{cases} 1 & \text{for } x^0 > 0, \\ 0 & \text{for } x^0 = 0, \\ -1 & \text{for } x^0 < 0, \end{cases}$$

we can write

$$\xi = \frac{1}{4\sqrt{p^0 p'^0}} \left[ \mathcal{D}(k^0, k', k^0, k; q^0, q) + i \mathcal{A}(k^0, k', k^0, k; q^0, q) \right], \quad (6.4)$$

where  $\mathcal{D}$  and  $\mathcal{A}$  are real functions of the invariant  $s = (k + k')^2$  and  $t = (k - k')^2$ . Introduction of a complete set of states and working in the system in which  $k + k' = 0$  then gives



$$q = p + \frac{1}{2}k - k',$$

$$\mathcal{D} = -i \int d^4x e^{iq \cdot x} \langle k' | \varepsilon(x) [j(-\frac{x}{2}), j^T(\frac{x}{2})] | k \rangle, \quad (6.8)$$

and  $\mathcal{A}$  is the same expression without the factor  $i\varepsilon(x)$ .  
Then

$$\mathcal{E}^* = \frac{1}{4\sqrt{p^0 p'^0}} [\mathcal{D}(k^0, k; k'^0, k'; q^0, q) - i\mathcal{A}(k^0, k; k'^0, k'; q^0, q)]. \quad (6.5)$$

If we assume that  $q$  is an arbitrary four-vector, then since  $\mathcal{D}$  and  $\mathcal{A}$  are invariant under Lorentz transformations, they must be functions of

$$q^2, \quad (k+k').q, \quad (k'-k).q, \quad (k'+k).(k'-k). \quad (6.6)$$

From (6.4) and (6.5) we see then that  $\mathcal{D}$  and  $\mathcal{A}$  will be real if they are even functions of the last two variables which are equal to zero when we have conservation of energy and momentum.

If we apply time-reversal to (6.3) we get

$$(\mathcal{E})^* = \frac{1}{4\sqrt{p^0 p'^0}} [\mathcal{D}(k^0, -k; k'^0, -k'; q^0, -q) + i\mathcal{A}(k^0, -k; k'^0, -k'; q^0, -q)]. \quad (6.7)$$

Comparison of (6.4) and (6.7) then shows that  $\mathcal{D}$  and  $\mathcal{A}$  are indeed even in the last two variables of (6.6). This means that the imaginary part of  $\mathcal{E}$  is  $\mathcal{A}$ . Introduction of a complete set of states and working in the system in which  $p + k = 0$  then gives

APPENDIX 1 : CONTRIBUTION FROM DIRECT GRAPHS

$$g_m \xi(k'; k) = \sum_{k_n^0 = k^0} \xi(k'; k_n) \xi^+(k; k_n). \quad (6.8)$$

Recalling our definition of the  $T^{(1)}$  matrix elements, we see that the same manipulation can be carried out for the other expression in it, and the result is the analogue of (6.2).

The same procedure can be carried out in the case of fermion-fermion scattering without difficulty.

We have not gone very far with the possible analytic properties of our T-matrix elements in Q.F.T. We look forward to the time when that is done so that we can test our definition of the potential between two particles.

with

$$S = \tau \alpha \omega + (V - \alpha \omega)$$

$$\lambda = (E^2 - \mu^2) \omega + \alpha \omega$$

$$K = \frac{1}{2}(P^2 - \mu^2)$$

$$P' = P + \alpha \omega$$

$$P = \frac{1}{2}(P'^2 - \mu^2)$$

$$E = \frac{1}{2}(E'^2 - \mu^2)$$

$$A' = -\frac{1}{2} \frac{P'^2 - \mu^2}{P' - \mu} \quad (A1.2)$$

$$A = -\frac{1}{2} \frac{P^2 - \mu^2}{P - \mu}$$

$$B = -\frac{1}{2} \frac{P^2 - \mu^2}{P + \mu}$$

# APPENDIX 1 : CONTRIBUTION FROM DIRECT GRAPHS

We follow closely the method of attack used by Charap & Tausner<sup>(22)</sup>. Application of the equations

$$q_2 = k' - k + q_1,$$

and

$$q = \frac{1}{2}(q_1 + q_2),$$

reduces (5.10) to the form

$$\frac{i q_0^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} \sum_{j,l=1}^3 \sum_{\mu,\nu=0}^3 \left[ \bar{u}(p') \gamma_\mu \tau_j \tau_l u(p) \bar{u}(k') \gamma_\nu \tau_j \tau_l u(k) I^{\mu\nu} - (p \leftrightarrow k) \right], \quad (A1.1)$$

where

$$I^{\mu\nu} = \int_{-\infty}^{\infty} d^4 q \frac{(q+t)^\mu (q+t)^\nu}{\{(N+q)^2 + m^2 - i\varepsilon\} \{(P-q+t'-t)^2 + m^2 - i\varepsilon\} \{(q-t)^2 + \mu^2 - i\varepsilon\} \{(q+t)^2 + \mu^2 - i\varepsilon\}}$$

$$= \frac{3}{2} \int_{-\infty}^{\infty} d^4 q \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 d\omega \frac{\omega(1-\omega)(q+t)^\mu (q+t)^\nu}{(q^2 - 2S \cdot q + \Lambda)^4},$$

with

$$S = tx\omega + (K + Cy)(1-\omega),$$

$$\Lambda = (t^2 + \mu^2)\omega + (A - By)(1-\omega) - i\varepsilon,$$

$$K = \frac{1}{2}(P' - N),$$

$$C = \frac{1}{2}(P' + N),$$

$$P' = P + t' - t,$$

$$P = \frac{1}{2}(p' + p),$$

$$N = \frac{1}{2}(k' + k),$$

$$t = \frac{1}{2}(k' - k),$$

$$t' = \frac{1}{2}(p - p'),$$

$$\Delta'^2 = t'^2 - (t' - t) \cdot (2P + t' - t) = t'^2 - 2(k'^0 - k^0)k^0,$$

$$A = -\frac{1}{2}(t^2 + \Delta'^2) = -t^2 + k^0(k'^0 - k^0),$$

$$B = -\frac{1}{2}(t^2 - \Delta'^2) = -k^0(k'^0 - k^0).$$

(A1.2)

Symmetric integration then gives

$$\begin{aligned} I^{\mu\nu} &= \frac{3}{2} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 d\omega \int_{-\infty}^{\infty} dq \frac{\omega(1-\omega)[q^\mu + q^\nu + (S+t)^\mu(S+t)^\nu]}{(q^2 + \Lambda - S^2)^4} \\ &= \frac{3}{2} \frac{\pi^2 i}{12} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 d\omega \omega(1-\omega) \left[ \frac{q^{\mu\nu}}{\Lambda - S^2} + \frac{2(S+t)^\mu(S+t)^\nu}{(\Lambda - S^2)^2} \right] \quad (A1.3) \end{aligned}$$

On the energy-shell this expression should coincide with the one obtained by Charap & Tausner who seemed to have made a mistake in obtaining  $\delta_{\mu\nu}$  in place of our  $q^{\mu\nu}$ .

$$\Lambda - S^2 \doteq \mu^2 \omega + \omega^2 t^2 (1-x^2) - (1-\omega)^2 (\eta^2 - m^2 y^2) + 2 \underline{u} \cdot \underline{t} (1-\omega)(1+x\omega) - i\varepsilon. \quad (A1.4)$$

$$\begin{aligned} &\sum_{\mu=0}^3 \bar{u}(k') \gamma^\mu u(k) \bar{u}(p') \gamma_\mu u(p) \\ &\doteq \left(1 + \frac{t^2}{m^2}\right) \mathcal{O}_1 + \frac{3}{2m^2} \mathcal{O}_2 + \frac{1}{4m^4} \mathcal{O}_3 + \frac{1}{m^2} \mathcal{O}_4 - \frac{t^2}{m^2} \mathcal{O}_5. \quad (A1.5) \end{aligned}$$

$$\begin{aligned} &\bar{u}(k') \gamma_\nu (S+t) u(k) \bar{u}(p') \gamma_\nu (S+t) u(p) 4m^2 (m+k^0)(m+k'^0) \\ &= -(1-\omega)(4m^2 + \underline{t}^2 + 3\underline{u}^2)^2 \left[ (2x\omega + 1 + \omega) \frac{(\underline{u} \cdot \underline{t})^2}{m^2} + (1-\omega) k^{02} y^2 \right] + (1-\omega) (k'^0 + k^0 + 2m)^2 \underline{u}^2 \\ &\times \left[ (1-\omega) \underline{u}^2 - 2(1+x\omega) \underline{u} \cdot \underline{t} \right] - i(1-\omega) (\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}) \cdot \underline{u} \wedge \underline{t} \left[ -2(1-\omega) k^{02} y^2 (4m^2 + \underline{t}^2 + 3\underline{u}^2) \right. \\ &\left. + \{ (1-\omega) \underline{u}^2 - (1+\omega + 2x\omega) \underline{u} \cdot \underline{t} \} (k'^0 + k^0 + 2m)^2 + 8(\underline{u} \cdot \underline{t})^2 (1+\omega + 2x\omega) \right] \\ &+ \underline{\sigma}^{(1)} \cdot \underline{u} \wedge \underline{t} \underline{\sigma}^{(2)} \cdot \underline{u} \wedge \underline{t} (1-\omega) \left[ 4k^{02} y^2 - (k^0 + k'^0 + 2m)^2 \right] + \\ &\times \left[ (1-\omega) \underline{\sigma}^{(1)} \cdot \underline{u} \wedge \underline{t} + (1+\omega + 2x\omega) \underline{\sigma}^{(2)} \cdot \underline{u} \wedge \underline{t} \right] + \text{terms odd in } y. \quad (A1.6) \end{aligned}$$

We see then (A1.6) is not symmetric with respect to  $\underline{\sigma}^{(1)} \cdot \underline{u} \wedge \underline{t}$  and

$\underline{\sigma}^{(2)} \cdot \underline{u} \wedge \underline{t}$ . Let us turn to (5.12). If we put

$$q = q_1 + q_2,$$



$$p' = q_z + p - q_1,$$

(5.12) can be cast into the form

$$\frac{i q^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} \sum_{j,l=1}^3 \sum_{\mu,\nu=0}^4 \left[ \bar{u}(p') \gamma_\mu \tau_j \tau_l u(p) \bar{u}(k') \gamma_\nu \tau_j \tau_l u(k) I'^{\mu\nu} - (p \leftrightarrow k) \right], \quad (A1.7)$$

with

$$\begin{aligned} I'^{\mu\nu} &= \int d^4 q \frac{(q+t')^\mu (q+t')^\nu}{\{ (N+q+t'-t)^2 + m^2 - i\varepsilon \} \{ (P-q)^2 + m^2 - i\varepsilon \} \{ (q+t')^2 + \mu^2 - i\varepsilon \} \{ (q-t')^2 + \mu^2 - i\varepsilon \}} \\ &= \frac{3}{2} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 d\omega \omega(1-\omega) \int_{-\infty}^{\infty} d^4 q \frac{(q+t')^\mu (q+t')^\nu}{(q^2 - 2S' \cdot q + \Lambda')^4} \\ &= \frac{3}{2} \frac{\pi^2 i}{12} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 d\omega \omega(1-\omega) \left[ \frac{g^{\mu\nu}}{\Lambda' - S'^2} + \frac{2(S'+t')^\mu (S'+t')^\nu}{(\Lambda' - S')^2} \right], \quad (A1.8) \end{aligned}$$

where  $N' = N + t' - t, \quad K' = \frac{1}{2}(P - N'),$

$$S' = t' x \omega + (K' + C y)(1-\omega), \quad (A1.9)$$

$$\Lambda' = (t'^2 + \mu^2)\omega + (A + B y)(1-\omega) - i\varepsilon,$$

and the rest are defined as in (A1.2).

$$\Lambda' - S' \doteq \mu^2 \omega + \omega^2 t'^2 (1-x^2) - (1-\omega)^2 (\eta^2 - k^2 y^2) + 2 \underline{u} \cdot t' (1-\omega)(1+x\omega),$$

which is exactly the same as (A1.4).

$$\bar{u}(k')(s'+t') \cdot \gamma u(k) \bar{u}(p')(s'+t') \cdot \gamma u(p) 4m^2 (m+k^0)(m+k'^0)$$

is approximately the same as (A1.6) except that  $\underline{\sigma}^{(1)}$  &  $\underline{\sigma}^{(2)}$  are interchanged in the last square bracket.

The averaging of (A1.1) and (A1.7) means the averaging of (A1.6) and (A1.9); the latter leads to the last square bracket in (A1.6) being changed to

$$(1 + \chi\omega)(\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}), \underline{u} \wedge \underline{t}.$$

Dividing through this average by

$$4m^2(m + k^0)(m + k'^0),$$

we then get approximately

$$\begin{aligned} & (1-\omega) \left[ -m^2 y^2 (1-\omega) + (1-\omega) \frac{\underline{t}^2 \underline{u}^2}{m^2} - (1+\chi\omega) \frac{\eta^2 \underline{u} \cdot \underline{t}}{m^2} + (1+\omega+2\chi\omega) \frac{(\underline{u} \cdot \underline{t})^2}{m^2} \right] \mathcal{Q}_1 + \\ & + (1-\omega) \left[ - (1-\omega) \frac{y^2}{2} + \frac{\underline{u}^2}{m^2} - (1+\chi\omega) \frac{\underline{u} \cdot \underline{t}}{m^2} \left( 1 - \frac{\underline{u}^2}{2m^2} \right) + (1+\omega+2\chi\omega) \frac{(\underline{u} \cdot \underline{t})^2}{2m^2} \right] \mathcal{Q}_2 + \\ & + (1-\omega)^2 \left[ \frac{y^2}{4m^2} - \frac{1}{m^2} \right] \mathcal{Q}_3 + \text{terms odd in } y. \end{aligned} \quad (\text{A1.9})$$

Substitution of (A1.5) and (A1.9) then leads to (5.13).

# APPENDIX 2: CONTRIBUTION FROM THE "CROSSED" GRAPHS

Putting  $q = \frac{1}{2}(q_1 + q_2)$  we get from (5.15)

$$\frac{i q_0^4}{(2\pi)^7} \frac{m^2}{k^0 k'^0} (3 + 2 \underline{t}^{(1)} \cdot \underline{t}^{(2)}) \sum_{\mu, \nu=0}^3 \left[ \bar{u}(p') \gamma_\mu u(p) \bar{u}(k') \gamma_\nu u(k) J^{\mu\nu} - (p \leftrightarrow k) \right], \quad (A2.1)$$

where

$$\begin{aligned} J^{\mu\nu} &= \int d^4 q \frac{(t+q)^\mu (t+q)^\nu}{\{(p'-q)^2 + m^2 - i\varepsilon\} \{(N-q)^2 + m^2 - i\varepsilon\} \{(q+t)^2 + \mu^2 - i\varepsilon\} \{(q-t)^2 + \mu^2 - i\varepsilon\}} \\ &= \frac{3}{2} \int_{-\infty}^{\infty} d^4 q \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \frac{\omega(1-\omega) (q+t)^\mu (q+t)^\nu}{(q^2 - 2R \cdot q + \Lambda)^4} \\ &= \frac{3}{2} \frac{\pi^2 i}{12} \int_{-1}^1 dx \int_{-1}^1 dy \int_0^1 dw \omega(1-\omega) \left[ \frac{q^{\mu\nu}}{\Lambda - R^2} + \frac{2(R+t)^\mu (R+t)^\nu}{(\Lambda - R^2)^2} \right], \quad (A2.2) \end{aligned}$$

with

$$R = tx\omega + (Ky + C)(1-\omega),$$

and all the other symbols as defined in (A1.2).

$$\begin{aligned} \Lambda - R^2 &\doteq \mu^2 \omega + \omega^2 t^2 (1-x^2) - (1-\omega)^2 [(1-y^2) \underline{t}^2 + \eta^2 y^2 - k^2] \\ &\quad + k^0 (k'^0 - k^0) (1+y)(1-\omega) \\ &\doteq \mu^2 \omega + \omega^2 \underline{t}^2 (1-x^2) + (1-\omega)^2 m^2, \end{aligned}$$

(A2.3)

$$\bar{u}(p') \gamma_\cdot (R+t) u(p) \bar{u}(k') \gamma_\cdot (R+t) u(k)$$

$$\doteq -m^2 (1-\omega)^2 \mathcal{O}_1 - \frac{1}{2} (1-\omega)^2 \mathcal{O}_2 + \left(\frac{1}{4} - y^2\right) \frac{1}{m^2} \mathcal{O}_3$$

+ terms odd in  $y$ .

(A2.4)

From (A1.5), (A2.2), (A2.3) and (A2.4) we see the coefficients of  $\mathcal{O}_i$  in (A2.1) are energy independent, and therefore (A2.1) must be the same as the corresponding expression from (5.11). We thus get (5.16).

Equation (5.16) is essentially the same as that obtained by Charap and Tausner except  $V_3^B$ ; the two expressions for  $V_3^B$  differ by a minus sign throughout. The difference in the term involving  $U$  is due, we believe, to an error in Charap & Tausner's calculation. That in the term involving  $F$  is because there is a contribution to (A1.5) off the energy-shell, namely,

$$\begin{aligned} & \underline{u} \cdot \underline{t} [\underline{\sigma}^{(1)} \cdot \underline{u} \underline{\sigma}^{(2)} \cdot \underline{t} + \underline{\sigma}^{(2)} \cdot \underline{u} \underline{\sigma}^{(1)} \cdot \underline{t}] \\ &= \underline{\sigma}^{(1)} \cdot \underline{u} \wedge \underline{t} \underline{\sigma}^{(2)} \cdot \underline{u} \wedge \underline{t} + \underline{t}^2 \underline{\sigma}^{(1)} \cdot \underline{u} \underline{\sigma}^{(2)} \cdot \underline{u} + \underline{u}^2 \underline{\sigma}^{(1)} \cdot \underline{t} \underline{\sigma}^{(2)} \cdot \underline{t} \end{aligned}$$

(apart from some trivial factors which is not important for the moment) which vanishes on the energy shell. In the process of approximation that led to (A1.5) we threw away the last two terms because they are small compared with the other expressions in the coefficients of  $\mathcal{O}_4$  and  $\mathcal{O}_6$  in (A1.5). This same cause accounts for the difference in  $G_3^A$  and hence  $V_3^A$ .



APPENDIX 3: CONTRIBUTION FROM THE ITERATED ONE-MESON-EXCHANGE GRAPHS.

From (5.19a), we get

$$J = -\frac{\pi}{8m} \int_{-\infty}^{\infty} d^3 p_z \int_{-1}^1 dx \int_0^1 d\omega \omega \left[ p_z^2 + \{ \mu^2 + \underline{u}^2 + \underline{t}^2 - 2p_z \cdot \underline{u} - 2x \underline{t} \cdot (\underline{p}_z - \underline{u}) \} \omega \right. \\ \left. - \{ \underline{u}^2 + \underline{t}^2 - 2\underline{u} \cdot \underline{t} \} (1-\omega) - i\varepsilon \right]^{-3} \left[ -\{ (\underline{p}_z - \underline{u})^2 - \underline{t}^2 \}^2 - 2i(\sigma^{(1)} + \sigma^{(2)}) \cdot (\underline{p}_z - \underline{u}) \wedge \underline{t} \right. \\ \left. \times \{ (\underline{p}_z - \underline{u})^2 - \underline{t}^2 \} + 4 \sigma^{(1)} \cdot (\underline{p}_z - \underline{u}) \wedge \underline{t} \sigma^{(2)} \cdot (\underline{p}_z - \underline{u}) \wedge \underline{t} \right],$$

Using the transformation

$$q = p_z - (\underline{u} + x \underline{t}) \omega,$$

we get

$$J = -\frac{\pi}{8m} \int_{-\infty}^{\infty} d^3 q \int_{-1}^1 dx \int_0^1 d\omega \omega (q^2 + a - i\varepsilon)^{-3} \left[ -\{ (q - \underline{u} + (\underline{u} + x \underline{t}) \omega)^2 - \underline{t}^2 \}^2 \right. \\ \left. - 2i(\sigma^{(1)} + \sigma^{(2)}) \cdot \{ -\underline{u} \wedge \underline{t} + q \wedge \underline{t} + \omega(\underline{u} + x \underline{t}) \wedge \underline{t} \} \{ (q - \underline{u} + (\underline{u} + x \underline{t}) \omega)^2 - \underline{t}^2 \} \right. \\ \left. + 4 \sigma^{(1)} \cdot \{ (-\underline{u} + q + \omega(\underline{u} + x \underline{t})) \wedge \underline{t} \} \sigma^{(2)} \cdot \{ (q - \underline{u} + \omega(\underline{u} + x \underline{t})) \wedge \underline{t} \} \right].$$

Routine simplification and symmetry integration then change the numerator to

$$q^4 + \frac{10}{3} q^2 \{ (-\underline{u} + \underline{u} \omega + x \omega \underline{t})^2 - \frac{3}{5} \underline{t}^2 \} + \{ (-\underline{u} + \underline{u} \omega + x \omega \underline{t})^2 - \underline{t}^2 \}^2 - \\ - i(\sigma^{(1)} + \sigma^{(2)}) \cdot \underline{u} \wedge \underline{t} (1-\omega) \{ \frac{10}{3} q^2 + 2(-\underline{u} + \underline{u} \omega + x \omega \underline{t})^2 - 2 \underline{t}^2 \} - \\ - 4 \sigma^{(1)} \cdot \underline{u} \wedge \underline{t} \sigma^{(2)} \cdot \underline{u} \wedge \underline{t} (1-\omega)^2 - \frac{4}{3} q^2 \underline{t}^2 \sigma^{(1)} \cdot \sigma^{(2)} + \frac{4}{3} q^2 \sigma^{(1)} \cdot \underline{t} \sigma^{(2)} \cdot \underline{t} \\ = [q^4 + \frac{10}{3} q^2 \{ \underline{u}^2 (1-\omega)^2 - 2x \omega (1-\omega) \underline{u} \cdot \underline{t} + x^2 \omega^2 \underline{t}^2 - (\underline{u}^2 - \underline{t}^2)(1-\omega) - \frac{3}{5} \underline{t}^2 \} \\ + \{ \underline{u}^2 (1-\omega)^2 - 2x \omega (1-\omega) \underline{u} \cdot \underline{t} + x^2 \omega^2 \underline{t}^2 - \underline{t}^2 \} \{ \underline{u}^2 (1-\omega)^2 - 2x \omega (1-\omega) \underline{u} \cdot \underline{t} + x^2 \omega^2 \underline{t}^2 - \underline{t}^2 \\ - 2(\underline{u}^2 - \underline{t}^2)(1-\omega) \}] \mathcal{O}_1 + [\frac{10}{3} q^2 + 2 \{ \underline{u}^2 (1-\omega)^2 - 2x \omega (1-\omega) \underline{u} \cdot \underline{t} + x^2 \omega^2 \underline{t}^2 - \underline{t}^2 \} \\ \times (1-\omega) \mathcal{O}_2 - 4(1-\omega)^2 \mathcal{O}_3 + \frac{4}{3} q^2 \mathcal{O}_4 - \frac{4}{3} q^2 \underline{t}^2 \mathcal{O}_5].$$

The angular integrations can be performed and lead to

$$J = \sum_{i=1}^6 J_i \mathcal{Q}_i, \quad (A3.2)$$

with  $J_i = \frac{\pi^2}{2m} \int_{-\infty}^{\infty} dq q^2 \int_{-1}^1 dx \int_0^1 d\omega \omega (\text{numerator})_i (q^2 + a - i\varepsilon)^{-3}$  (A3.3)

the (numerators)<sub>i</sub> being the coefficients of  $\mathcal{Q}_i$  in (A3.2).

A complicated series of integrations by parts then lead from (A3.1) and (A3.2) to (5.20). We will not show them in detail here, since they are mostly routine except in the reductions of  $J_1$  and  $J_2$  when we use the following two Lemmas<sup>\*</sup>:-

Lemma 1.

$$\begin{aligned} & \therefore \int_0^{\infty} dq \int_0^1 d\omega \omega^n (q^2 + a - i\varepsilon)^{-1} \\ &= \int_0^{\infty} dq (q^2 + a - i\varepsilon)^{-1} \omega^n \Big|_{\omega=0}^{\omega=1} - \int_0^{\infty} dq \int_0^1 d\omega \omega \frac{d}{d\omega} \{ \omega^{n+1} (q^2 + a - i\varepsilon)^{-1} \} \\ &= \frac{\pi}{2} \frac{1}{a_1} - (n + \frac{1}{2}) \int_0^{\infty} dq \int_0^1 d\omega \omega^n (q^2 + a - i\varepsilon)^{-1} \\ & \quad + \int_0^{\infty} dq \int_0^1 d\omega \omega^n (q^2 + a - i\varepsilon)^{-2} \{ \omega^2 \underline{t}^2 (1 - x^2) + \eta^2 (1 - \omega^2) - 2 \underline{u} \cdot \underline{t} (1 + x\omega^2) \}, \\ & \therefore, \int_0^{\infty} dq \int_0^1 d\omega \omega^n (q^2 + a - i\varepsilon)^{-2} \{ \omega^2 \underline{t}^2 (1 - x^2) + \eta^2 (1 - \omega^2) - 2 \underline{u} \cdot \underline{t} (1 + x\omega^2) \} \\ &= -\frac{\pi}{2} \frac{1}{a_1} + (n + \frac{3}{2}) \int_0^{\infty} dq \int_0^1 d\omega \omega^n (q^2 + a - i\varepsilon)^{-1}, \end{aligned}$$

with  $a_1 = \mu^2 + \underline{t}^2 (1 - x^2),$

<sup>\*</sup> We would like to thank Dr. J.M. Charap for sending us the details of the corresponding two lemmas he and Dr. M.J. Tausmer used in their paper<sup>(22)</sup>. Our two lemmas are mere adaptations of their two.

Lemma 2.

In a similar manner we can prove that

$$\int_0^\infty dq q^2 \int_0^1 w dw w^n (q^2 + a - i\varepsilon)^{-2} \{ \frac{1}{2} w^2 (1-x^2) + \eta^2 (1-w^2) - 2u \cdot \frac{1}{2} (1+xw^2) \}$$

$$= \frac{\pi}{2} a_1 - Q \left\{ 1 + \frac{1}{2(n+2)} \right\} + (n + \frac{5}{2}) \int_0^\infty dq q^2 \int_0^1 w dw w^n (q^2 + a - i\varepsilon)^{-1},$$

where  $Q$  is an arbitrarily large positive number.

$$= \frac{1}{4} \int_0^\infty dw \frac{w}{a - i\varepsilon} \left[ \frac{2\pi i}{-2i(a - i\varepsilon)^{1/2}} \right]$$

$$= -\frac{\pi}{4} \int_0^\infty dw w (a - i\varepsilon)^{-1/2},$$

where we take the branch of the square root in which the function is real if its argument approaches the positive real axis from above.

Write

$$a(w) = -bw^2 + cw - d,$$

with

$$b = \frac{1}{2} - \eta^2 (1-x^2) + 2x \frac{1}{2} \cdot \frac{1}{2},$$

$$c = \eta^2 + 2\eta^2 - 2\frac{1}{2} \cdot \frac{1}{2} (1-x),$$

$$d = \eta^2 - 2\frac{1}{2} \cdot \frac{1}{2}.$$

The roots of  $a(w) = 0$  are

$$w_0 = \frac{-c \pm \sqrt{c^2 - 4bd}}{-2b},$$

and they are real because

$$c^2 - 4bd = \eta^4.$$

APPENDIX 4: ESTIMATE OF CERTAIN INTEGRALS

$$\begin{aligned}
 \mathcal{I}_1 &= \int_0^\infty dq \int_0^1 \omega d\omega (q^2 + a - i\varepsilon)^{-2} \\
 &= \frac{1}{2} \int_0^1 d\omega \omega \left[ \frac{2q}{4(a-i\varepsilon)(q^2 + a - i\varepsilon)} \right]_{q=-\infty}^{q=\infty} + \frac{2}{4(a-i\varepsilon)} \int_{-\infty}^\infty \frac{dq}{q^2 + a - i\varepsilon} \\
 &= \frac{1}{4} \int_0^1 d\omega \frac{\omega}{a - i\varepsilon} \left[ \frac{2\pi i}{-2i(a-i\varepsilon)^{1/2}} \right] \\
 &= -\frac{\pi}{4} \int_0^1 d\omega \omega (a - i\varepsilon)^{-3/2},
 \end{aligned}$$

where we take the branch of the square root in which the function is real if its argument approaches the positive real axis from above.

Write

$$a(\omega) = -b\omega^2 + c\omega - d,$$

with

$$b = \eta^2 - t^2(1-x^2) + 2xt.u,$$

$$c = \mu^2 + 2\eta^2 - 2t.u(1-x),$$

$$d = \eta^2 - 2t.u,$$

The roots of  $a(\omega) = 0$  are

$$\omega_0 = \frac{-c \pm \sqrt{c^2 - 4bd}}{-2b},$$

and they are real because

$$c^2 - 4bd \geq \mu^4.$$



We will therefore change the integration over real  $\omega$  into one over complex  $\omega$ . Thus

$$\begin{aligned} & + \frac{\pi}{4} \int_0^1 d\omega \omega (a - i\varepsilon)^{-3/2} = - \frac{\pi}{2} \int_{0-i\varepsilon}^{1-i\varepsilon} d\omega \omega a^{3/2} \\ & = - \frac{\pi}{4} \frac{-2c+4d}{c^2-4bd} \frac{1}{\sqrt{\mu^2+t^2(1-x^2)}} + \frac{\pi}{4} \frac{2i\sqrt{d}}{c^2-4bd}. \end{aligned}$$

Both the real and imaginary part is less than  $2/\mu^3$  in magnitude and thus

$$|g_1| < \pi/\sqrt{2}\mu^3.$$

$$\begin{aligned} g_2 &= \int_0^\infty dq \int_0^1 d\omega \omega^2 (q^2 + a - i\varepsilon)^{-2} \\ &= - \frac{\pi}{4} \int_0^1 d\omega \omega^2 (a - i\varepsilon)^{-3/2} \\ &= - \frac{\pi}{4} \int_0^1 d\omega (a - i\varepsilon)^{-1/2} \left[ -\frac{1}{b} + \frac{c\omega - d}{b(a - i\varepsilon)} \right] \\ &= - \frac{\pi}{4} \left[ \frac{1}{b^{3/2}} \left\{ \cos^{-1} \frac{c-2b}{\sqrt{c^2-4bd}} - \cos^{-1} \frac{c}{\sqrt{c^2-4bd}} \right\} - \right. \\ &\quad \left. - \frac{c}{b} \frac{2c\omega - 4d}{(4bd - c^2)\sqrt{a-i\varepsilon}} \Big|_{\omega=0}^{\omega=1} - \frac{d}{b} \frac{-4b\omega + 2c}{(4bd - c^2)\sqrt{a-i\varepsilon}} \Big|_{\omega=0}^{\omega=1} \right] \\ &= - \frac{\pi}{4} \left[ \frac{1}{b^{3/2}} \left\{ \cos^{-1} \frac{c-2b}{\sqrt{c^2-4bd}} - \cos^{-1} \frac{c}{\sqrt{c^2-4bd}} \right\} - \frac{1}{b\sqrt{c-b-d}} \times \right. \\ &\quad \left. \times \left\{ 2 - \frac{2d(c-2b)}{c^2-4bd} \right\} + \frac{2c\sqrt{d}}{ib(c^2-4bd)} \right]. \end{aligned}$$

Now provided  $|b|$  and  $|d|$  are less than or equal to  $\mu$ ,

$$\frac{4bd}{c^2} < \frac{4}{9}, \quad \frac{d}{c} < \frac{1}{2}, \quad \frac{b}{c-d} < \frac{1}{2},$$

and so

$$\begin{aligned} \cos^{-1} \frac{c}{\sqrt{c^2 - 4bd}} &\doteq \cos^{-1} \left( 1 + \frac{2bd}{c^2} + \frac{3}{8} \frac{(4bd)^2}{c^4} + \frac{5}{16} \frac{(4bd)^3}{c^6} \right) \\ &= 2i \sinh^{-1} \left( \frac{bd}{c^2} + \frac{3b^2d^2}{c^4} + 10 \frac{b^3d^3}{c^6} \right)^{1/2} \\ &\doteq 2i \left[ \frac{\sqrt{bd}}{c} \left( 1 + \frac{3bd}{2c^2} + 5 \frac{b^2d^2}{c^4} \right) - \frac{(bd)^{3/2}}{6c^3} \left( 1 + \frac{3bd}{2c^2} \right) \right] \\ &= 2i \frac{\sqrt{bd}}{c} \left( 1 + \frac{4bd}{3c^2} + \frac{19}{4} \frac{b^2d^2}{c^4} \right), \\ \cos^{-1} \frac{c-2b}{\sqrt{c^2 - 4bd}} &\doteq \cos^{-1} \left[ \left( 1 - \frac{2b}{c} \right) \left( 1 + \frac{2bd}{c^2} + \frac{3}{8} \frac{(4bd)^2}{c^4} + \frac{5}{16} \frac{(4bd)^3}{c^6} \right) \right] \\ &= \cos^{-1} \left[ 1 - \frac{2b}{c} + \frac{2bd}{c^2} + \frac{6b^2d^2}{c^4} - \frac{4b^2d}{c^3} + \frac{20b^3d^3}{c^6} - \frac{12b^3d^2}{c^5} \right] \\ &= 2 \sin^{-1} \left( \frac{b}{c} - \frac{bd}{c^2} - \frac{3b^2d^2}{c^4} + \frac{2b^2d}{c^3} - \frac{10b^3d^3}{c^6} - \frac{6b^3d^2}{c^5} \right) \\ &= 2 \left[ \frac{\sqrt{b(c-d)}}{c} + \frac{b^{3/2}(c^2 - 8d^2 + 4cd)}{6c^3\sqrt{c-d}} \right. \\ &\quad \left. + \frac{b^{5/2}d}{4c^5\sqrt{c-d}} (2c^2 + 7cd - 17d^2) \right], \end{aligned}$$

Thus from (5.26a),

$$\text{and } \frac{1}{\sqrt{c-d-b}} = \frac{1}{\sqrt{c-d}} \left( 1 + \frac{1}{2} \frac{b}{c-d} + \frac{3}{8} \frac{b^2}{(c-d)^2} \right).$$

$$\therefore, g_2 \doteq -\frac{\pi}{4} \left[ \frac{1}{\sqrt{c-d}} \left( \frac{2}{3c} - \frac{16d^2}{3c^3} + \frac{8d}{3c^2} \right) - i \frac{4d^{3/2}}{c^3} \right].$$

$$\therefore, |g_2| < \frac{\pi}{3\mu^3}.$$

$$\begin{aligned}
 \mathcal{G}_3 &= \int_0^1 d\omega \omega^3 \int_0^\infty dq (q^2 + a - i\varepsilon)^{-2} \\
 &= -\frac{\pi}{4} \int_0^1 d\omega \omega^3 (a - i\varepsilon)^{-3/2} \\
 &= -\frac{\pi}{4} \int_0^1 d\omega \left[ -\left(\frac{\omega}{b} + \frac{c}{b^2}\right) + \left\{ \frac{\omega}{b} \left( \frac{c^2}{b} - d \right) - \frac{cd}{b^2} \right\} \frac{1}{a - i\varepsilon} \right] (a - i\varepsilon)^{-1/2} \\
 &= -\frac{\pi}{4} \left[ \left\{ \left( \frac{c^2}{b^2} - \frac{d}{b} \right) \frac{-2c\omega + 4d}{(4bd - c^2)\sqrt{a - i\varepsilon}} - \frac{cd}{b^2} \frac{-4b\omega + 2c}{(4bd - c^2)\sqrt{a - i\varepsilon}} + \frac{\sqrt{a - i\varepsilon}}{b^2} \right\} \right. \\
 &\quad \left. - \left( \frac{c}{2b^2} + \frac{c}{b^2} \right) \int_0^1 d\omega (a - i\varepsilon)^{-1/2} \right] \\
 &= -\frac{\pi}{4} \left[ -\frac{1}{\sqrt{c-b-d}} \left\{ \frac{2d(2d-c)}{b(4bd-c^2)} + \frac{3c-3d-b}{b^2} \right\} + i\sqrt{d} \frac{-3c^2+8bd}{b^2(4bd-c^2)} \right. \\
 &\quad \left. + \frac{3c}{2b^{5/2}} \left\{ \cos^{-1} \frac{c-2b}{\sqrt{c^2-4bd}} - \cos^{-1} \frac{c}{\sqrt{c^2-4bd}} \right\} \right] \\
 &\doteq -\frac{\pi}{4} \left[ \frac{1}{\sqrt{c-d}} \left( \frac{d}{2c^2} - \frac{5}{8(c-d)} \right) + i \frac{d^{5/2}}{4c^4} \right].
 \end{aligned}$$

$$\therefore, |\mathcal{G}_3| < \frac{3\pi}{32\sqrt{2}} \mu^3.$$

Thus from (5.20a),

$$\begin{aligned}
 |L_s| &< \frac{\pi^2}{4m} \int_{-1}^1 dx (1+x)^2 |\mathcal{G}_3| \\
 &< \frac{3}{4\sqrt{2}} \frac{\pi^2}{4m\mu^3},
 \end{aligned}$$

Thus compared with  $X_c$ , we can ignore terms like

$$|G_s| < \frac{\pi^2}{4m} \int_{-1}^1 dx [|\mathcal{G}_1| + (1-x)|\mathcal{G}_2| + |x||\mathcal{G}_3|]$$

since they amount to only a few percent of the total.

$$< \frac{\pi^2}{4m} \left[ \frac{\pi}{\sqrt{2}\mu^3} + \frac{2\pi}{3\mu^3} + \frac{3\pi}{32\sqrt{2}\mu^3} \right] \doteq \frac{5\pi^2}{4m\mu^3},$$

$$|H_s| < \frac{\pi^2}{4m} \int_{-1}^1 dx (1+x) [|\mathcal{G}_2| + |\mathcal{G}_3|]$$

$$< \frac{\pi^2}{4m} \frac{1}{\mu^3} \left[ \frac{\pi}{3} + \frac{3\pi}{32\sqrt{2}} \right] < \frac{2\pi^2}{4m\mu^3}.$$

Now

$$X_c > \frac{\pi^2}{4m} \int_{-1}^1 dx \int_0^1 dw \omega \int_{m(1-w)}^{\infty} dq (q^2 + \mu^2 + t^2)^{-1}$$

$$= \frac{2\pi^2}{4m} \int_0^1 dw \frac{\omega}{\sqrt{\mu^2 + t^2}} \left[ \frac{\pi}{2} - \tan^{-1} \frac{m(1-w)}{\sqrt{\mu^2 + t^2}} \right]$$

$$= \frac{\pi^2}{2m} \int_0^1 dw \frac{\omega}{\sqrt{\mu^2 + t^2}} \cot^{-1} \frac{m(1-w)}{\sqrt{\mu^2 + t^2}}$$

$$= \frac{\pi^2}{4m} \left[ \frac{\pi}{2} (\mu^2 + t^2)^{-1/2} - \int_0^1 dw \frac{m\omega^2/2}{m^2(1-w)^2 + \mu^2 + t^2} \right]$$

$$= \frac{\pi^2}{8m} \left[ \pi(\mu^2 + t^2)^{-1/2} - m \left\{ \frac{\omega}{m^2} + \frac{1}{m^2} \log(m^2(1-w)^2 + \mu^2 + t^2) \right. \right.$$

Now for

$$\left. + \frac{m^2 - \mu^2 - t^2}{m^2} \frac{1}{m \sqrt{\mu^2 + t^2}} \tan^{-1} \frac{m(\omega-1)}{\sqrt{\mu^2 + t^2}} \right\}_0^1 \right]$$

$$\doteq \frac{\pi^2}{8m} \left[ \pi(\mu^2 + t^2)^{-1/2} - \frac{1}{m} \left\{ 1 - 2 \times 1.9 - \frac{m}{\sqrt{\mu^2 + t^2}} \left( \frac{\pi}{2} - 1.48 \right) \right\} \right]$$

and it can be seen that the latter has the highest value

$$\doteq \frac{\pi^2}{4m} \frac{\pi}{\sqrt{\mu^2 + t^2}},$$

at



Thus compared with  $X_c$ , we can ignore terms like

$$\frac{u \pm L_s}{2m^2}, \quad \frac{u^2}{2m^2} \frac{u \pm G_s}{2m^2}, \quad \frac{(u \pm L_s)^2}{2m^2} (G_s + H_s)$$

since they amount to only a few percents of the former.

$$\begin{aligned} X_s &= \frac{\pi^2}{16m} \int_{-1}^1 dx \int_0^1 dw \frac{w\pi}{\sqrt{a-i\epsilon}} \\ &= \frac{\pi^3}{16m} \int_{-1}^1 dx \left[ \frac{1}{b} (\sqrt{c-b-d} - i\sqrt{d}) \right. \\ &\quad \left. + \frac{c}{2b^{3/2}} \left( \cos^{-1} \frac{c-2b}{\sqrt{c^2-4bd}} - \cos^{-1} \frac{c}{\sqrt{c^2-4bd}} \right) \right] \\ &\doteq \frac{\pi^3}{16m} \int_{-1}^1 dx \left[ \frac{1}{\sqrt{c-d}} \left( -\frac{1}{3} + \frac{2d}{3c} - \frac{4d^2}{3c^2} \right) - i \frac{4d^{3/2}}{3c^2} \right]. \end{aligned}$$

$$\therefore |X_s| < \frac{\pi^3}{16m} \int_{-1}^1 dx \sqrt{2} = \frac{\pi^2}{4m} \frac{\pi}{\sqrt{2}} < X_c.$$

$$\begin{aligned} \therefore Y_c &< \frac{\pi^2}{2m} \int_{-1}^1 dx \int_0^1 dw w (1-w)^2 \int_{m(1-w)}^{\infty} \frac{dq / m^2 (1-w)^2}{\mu^2 w + t^2 w^2 (1-x^2) + q^2} \\ &= \frac{1}{m^2} X_c, \end{aligned}$$

$\therefore$  we can ignore  $\frac{u^2}{m^2} Y_c$  compared with  $X_c$ .

Now for  $|x| \leq 1$ ,  $0 < \omega < 1$ ,  $q \geq m(1-\omega)$ ,

$$\frac{1-\omega}{q^2 + \mu^2 \omega + (1-x^2)t^2 \omega^2} \leq \frac{1-\omega}{m^2(1-\omega)^2 + \mu^2 \omega + (1-x^2)t^2 \omega^2},$$

and it can easily be shown that the latter has its highest value at



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